Problem 1

Suppose that a class of 100 students consists of four subgroups, in the following proportion:

	Men	Women
Taking economics	17%	38%
Not taking economics	23%	22%

What is the chance that a randomly chosen student is:

(a) a woman?

(b) taking economics?

(c) a man or taking economics?

(d) a woman and taking economics?

Solution:

(a) Adding values in the column "Women", we acquire $\boxed{60\%}$.

(b) Adding values in the row "Taking economics", we acquire $\boxed{55\%}$.

(c) We have to find the union of sets "Men" and "Taking economics".

i) Adding values of all cells, except for the bottom right one, we acquire $\boxed{78\%}$.

ii) Using the union probability additive law:

$$\mathsf{P}(M \cup T) = \mathsf{P}(M) + \mathsf{P}(T) - \mathsf{P}(M \cap T) = 40\% + 55\% - 17\% = \boxed{78\%}$$

where M – set of "Men", and T – set of "Taking economics".

iii) Using De Morgan's law:

$$\mathsf{P}(M \cup T) = \mathsf{P}\left(\overline{\overline{M \cup T}}\right) = \mathsf{P}\left(\overline{\overline{M} \cap \overline{T}}\right) = \mathsf{P}\left(\overline{W \cap N}\right) = 100\% - 22\% = \boxed{78\%}$$

where W – set of "Women", and N – set of "Not taking economics".

All those solutions are basically different formalizations of the same idea.

(d) The intersection cell of labels "Women" and "Taking economics" is the top right one with value of $\boxed{38\%}$.

Problem 2

20 families live in a neighborhood of Wolfenstein Castle: 4 have 1 child, 8 have 2 children, 5 have 3 children, and 3 have 4 children. If we meet a local child near the castle, what are the probabilities $p_1,\ p_2,\ p_3,\ p_4$, that the child comes from a family with 1, 2, 3, 4 children?

Solution:

There are in total $4 \cdot 1 + 8 \cdot 2 + 5 \cdot 3 + 3 \cdot 4 = 47$ children. So, $\boxed{p_1 = \frac{4}{47},\ p_2 = \frac{16}{47},\ p_3 = \frac{15}{47},\ p_4 = \frac{12}{47}}$.

Problem 3

Suppose a word is picked at random from this sentence.

(a) What is the sample space of this random experiment?

(b) Find the probability that:

  i) the word has at least 4 letters;
  ii) the word contains at least 2 vowels;
  iii) the word contains at least 4 letters and at least 2 vowels.

Solution:

(a) Note that there are 10 words in that sentence. The sample space is a set of all the words in it.

(b) i) There are 7 words with at least 4 letters, so the probability is $\boxed{0.7}$

  ii) There are 4 words with at least 2 vowels, so the probability is $\boxed{0.4}$

  iii) There are 4 words with at least 4 letters and at least 2 vowels, so the probability is $\boxed{0.4}$

Problem 4

Find unions and intersections of the following events. In which case one event is a subset of the other?

(a) $A = \{1, 2, 5, 6\},\ B = \{1, 5\}$.

(b) $A = \{\text{Ann, Mary, Mike}\},\ B = \{\text{Tom, Mike, John}\}$.

(c) $A = \{\text{Moscow, London, Paris}\},\ B = \{\text{Paris, Berlin, Tokyo}\},\ C = \{\text{Tokyo, Rome}\}$.

Solution:

(a) $A \cup B = \{1, 2, 5, 6\},\ A \cap B = \{1, 5\}, B \subset A$

(b) $A \cup B = \{\text{Ann, Mary, Mike, Tom, John}\},\ A \cap B = \{\text{Mike}\}$

(c) $A \cup B \cup C = \{\text{Moscow, London, Paris, Berlin, Tokyo, Rome}\}, A \cap B \cap C = \varnothing$

Problem 5

In a group of students (none of whom are from DSBA) 25% smoke hookah, 60% drink alcohol, and 15% do both. What fraction of students have at least one of these bad habits?

Solution:

Let S = "Student smokes", A = "Student drinks alcohol"
We have $\mathsf{P}(S) = 0.25,\ \mathsf{P}(A) = 0.6,\ \mathsf{P}(S \cap A) = 0.15$, so $\mathsf{P}(S \cup A) = \mathsf{P}(A) + \mathsf{P}(S) - \mathsf{P}(S \cap A) = 0.25 + 0.6 - 0.15 = \boxed{70\%}$

Problem 6

In a group of 320 graduates of the School of Witchcraft and Wizardry (all of whom are either witches or wizards) only 160 went to college, but 100 of 170 wizards did. What is the probability that a witch chosen at random from the group of school graduates did not go to college?

Solution:

Since there are 170 wizards, there are $230 - 170 = 150$ witches. Also, only 100 wizards went to college, so $160 - 100 = 60$ witches went to college. So, $150 - 60 = 90$ witches didn't go to college, and the desired probability is $\boxed{\frac{90}{150}}$.

Problem 7

(a) If A and B are mutually exclusive events with probabilities of 0.6 and 0.2 respectively, then what is the probability of A or B occurring?

(b) If $\mathsf{P}(A) = 0.2,\ \mathsf{P}(B) = 0.3$, and $\mathsf{P}(A \cap B) = 0.1$, then what is $\mathsf{P}(A \cup B)$?

(c) If $\mathsf{P}(A) = 0.4,\ \mathsf{P}(B) = 0.5$, and $\mathsf{P}(A \cup B) = 0.7$, then what is $\mathsf{P}\left(\ \overline{A \cap B}\ \right)$?

Solution:

We know, that for any events, $A,\ B$

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B)$$

.

(a) Here $\mathsf{P}(A \cap B) = 0$, so $\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) = 0.6 + 0.2 = \boxed{0.8}$

(b) $\mathsf{P}(A \cup B) = 0.2 + 0.3 - 0.1 = \boxed{0.4}$

(c) $\mathsf{P}(A \cap B) = 0.4 + 0.5 - 0.7 = 0.2$. So $\mathsf{P}\left(\ \overline{A \cap B}\ \right) = 1 - \mathsf{P}(A \cap B) = \boxed{0.8}$

Problem 8

A survey of the houses in an old residential area found 30% with holes in the roof, 40% with broken windows, and 25% with the both problems.

(a) What is the proportion of houses with one or the other (or both) problems?

(b) What is the proportion of houses with holes in the roof but without broken windows?

(c) What is the proportion of houses with exactly one of these problems?

(d) What is the proportion of houses with none of these problems?

Solution:

Let H = "house is with holes in the roof", W = "house is with broken windows", $\mathsf{P}(H) = 0.3$, $\mathsf{P}(W) = 0.4$, $\mathsf{P}(H \cap W) = 0.25$.

(a) $\mathsf{P}(H \cup W) = \mathsf{P}(H) + \mathsf{P}(W) - \mathsf{P}(H \cap W) = 0.3 + 0.4 - 0.25 = \boxed{45\%}$

(b) $\mathsf{P}(H \setminus W) = \mathsf{P}(H) - \mathsf{P}(H \cap W) = 0.3 - 0.25 = \boxed{5\%}$

(c) $\mathsf{P}(H \cup W) - \mathsf{P}(H \cap W) = 0.45 - 0.25 = \boxed{20\%}$

(d) $1 - \mathsf{P}(H \cup W) = 1 - 0.45 = \boxed{55\%}$

Problem 9

Suppose that in families with three children births are independent, and the probability of a boy on each birth is 52 %. Use the table

Outcome	BBB	BBG	BGB	BGG	GBB	GBG	GGB	GGG
Probability	0.14	0.13	0.13	0.12	0.13	0.12	0.12	0.11

and find the chance that in a family of three children, there will be:

(a) exactly 2 girls;

(b) at least two girls;

(c) at least one child of each sex;

(d) the middle child being opposite in sex to the other two.

Solution:

Let B = "The child born is a boy", and G = "The child born is a girl". It is obvious that $\overline{B} = G$, and we are given that $\mathsf{P}(B) = 52\%$, so $\mathsf{P}(G) = 48\%$.

(a)
  i) Adding values of all cells, in which where are exactly two G's present, we acquire $12\% + 12\% + 12\% = \boxed{36\%}$
  ii) There are $C_3^2 = \dfrac{3!}{2! \cdot 1!}$ ways to choose the places for the girls(in the triplet of births), and the probability of each triplet is (because births are independent) $\mathsf{P}(B) \cdot \mathsf{P}(G) \cdot \mathsf{P}(G)$, so the answer is $3 \cdot 0.52 \cdot 0.48 \cdot 0.48 = 0.359424 = \boxed{36\%}$

(b)
  i) Adding values of all cells, in which where are at least two G's present, we acquire $12\% + 12\% + 12\% + 11\% = \boxed{47\%}$
  ii) We already know that the probability that in a family there are exactly two girls is $C_3^2 \cdot\ \mathsf{P}(B) \cdot\ \mathsf{P}(G) \cdot\ \mathsf{P}(G)$. Similarly, the probability of a family in which there are three girls is $C_3^3 \cdot\ \mathsf{P}(G)^3$. So, the answer is $3 \cdot 0.52 \cdot 0.48 \cdot 0.48 + 1 \cdot 0.48^3 = 0.470016 = \boxed{47\%}$

(c)
  i) Adding values of all cells, in which where are at least one G and at least one B present, we acquire $13\% + 13\% + 12\% + 13\% + 12\% + 12\% = \boxed{75\%}$
  ii) It's obvious that the desired probability is

$$1 - \mathsf{P}(\text{all in the family are boys}) - \mathsf{P}(\text{all in the family are girls}).$$

$\mathsf{P}(\text{all in the family are boys}) = \mathsf{P}(B)^3$, $\mathsf{P}(\text{all in the family are girls}) = \mathsf{P}(G)^3$, so the answer is $1 - \mathsf{P}(B)^3 - \mathsf{P}(G)^3 = 0.7488 = \boxed{75\%}$

(d) i) The only variants are BGB and GBG, so, by summing the probabilities, we have $13\% + 12\% = \boxed{25\%}$

ii) Similarly,because the births are independent, the probability of BGB is $\mathsf{P}(B) \cdot \mathsf{P}(G) \cdot \mathsf{P}(B)$, and the probability of BGB is $\mathsf{P}(G) \cdot \mathsf{P}(B) \cdot \mathsf{P}(G)$, by summing, we also have $\boxed{25\%}$

Problem 10

You take 3 cards from a deck of 36 cards. After you take each card you return it to the deck and shuffle the deck.

(a) What is the set of elementary outcomes and what are their probabilities?

(b) What is the probability to get (queen, king, ace) ?
(The first card is a queen, the second card is a king, the third card is an ace).

(c) What is the probability to get (king, king, ace) ?

Solution:

(a) Let X be the set of cards. Because we shuffle back, the set of elementary outcomes is $X \times X \times X$ (A set of all possible ordered triplets, so that all elements in the triplets $\in X$), and the probability of each triplet is $\left(\frac{1}{36}\right)^3$ (since the probability of getting a fixed card is $\frac{1}{36}$).

(b) there are 4 queens, 4 kings, 4 aces, and 36 cards, so, the probability of getting a queen (or a king or a ace, doesn't matter) when drawing a card is $\frac{4}{36}$, and so the probability to get (queen, king, ace) is $\left(\frac{4}{36}\right)^3 = \boxed{\left(\frac{1}{9}\right)^3}$

(c) It's obvious that the probability is the same as in b), since there are 4 kings and 4 queens, and we shuffle back.

Problem 11

You take 3 cards from a deck of 36 cards. After you take each card you do not return it to the deck. Answer (a), (b), (c) from the previous problem.

Solution:

(a) Let X be the set of cards. The set of elementary outcomes is

$$\Omega = \{(x, y, z) \mid x, y, z \in X, x \neq y \neq z\}$$

(all ordered triplets of X, so that all elements in the triplets are distinct, because we don't return the cards). And the probability of each triplet is $\frac{1}{36} \cdot \frac{1}{35} \cdot \frac{1}{34}$ (first there are 36 possible cards to take, then 35, and then 34).

(b) $\mathsf{P}((\text{queen, king, ace})) = \boxed{\frac{4 \cdot 4 \cdot 4}{36 \cdot 35 \cdot 34}}$
(because there are 4 queens, 4 kings, and 4 aces, and after taking a card the size of the deck decreases by one).

(c) $\mathsf{P}((\text{king, king, ace})) = \boxed{\frac{4 \cdot 3 \cdot 4}{36 \cdot 35 \cdot 34}}$
(because first there are 4 ways to choose the king, then 3 (we already have taken one), and then 4 ways to choose the ace).

Problem 12

Suppose we roll a red die and a green die. What is the probability that the number on the red die is larger than the number on the green die?

Solution:

Let A be the number on the red die, B – the number on the green die. Note, that $\mathsf{P}(A > B) = \mathsf{P}(A < B)$ (because of symmetry, the dies are equal). And $\mathsf{P}(A = B) = 6 \cdot \frac{1}{36} = \frac{1}{6}$ (for each value of the dice from 1 to 6 the probability that it is on both dices is 1/36).

So, $\mathsf{P}(A > B) = \frac{1 - \frac{1}{6}}{2} = \boxed{\frac{5}{12}}$.

Problem 13

Two dice are rolled. Find the probability that

(a) the two numbers will differ by 1 or less;

(b) the maximum of the two numbers will be 5 or larger.

Solution:

(a) Let A be the number on the first dice, B – the number on the second dice. Obviously, $\mathsf{P}\left(|A-B| \leq 1|\right) = \mathsf{P}\left(|A-B| = 0|\right) + \mathsf{P}\left(|A-B| = 1\right)$. We know that $\mathsf{P}(A = B) = 1/6$. Also, it is obvious, because of symmetry, that $\mathsf{P}(A = B+1) = \mathsf{P}(B = A+1)$.

$$\mathsf{P}(A = B+1) = \frac{1}{6} \cdot \left(0 + \frac{1}{6} + ... + \frac{1}{6}\right) = \frac{5}{36}.$$

So the desired probability is

$$\mathsf{P}(|A-B| \leq 1|) = \frac{1}{6} + 2 \cdot \frac{5}{36} = \boxed{\frac{4}{9}}.$$

(b) Event $\max(A, B) \geq 5$ is opposite to the event when neither of dice hit 5 or 6. It means that the probability is following:

$$\mathsf{P}\left(\max(A, B) \geq 5\right) = 1 - \mathsf{P}(A < 5 \cap B < 5) \overset{\text{ind}}{=} 1 - \mathsf{P}(A < 5) \cdot \mathsf{P}(B < 5) = 1 - \frac{2}{3} \cdot \frac{2}{3} = \boxed{\frac{5}{9}}.$$

Problem 14

In Galileo's time people thought that when three dice were rolled, a sum of 9 points and a sum of 10 points had the same probability, since each could be obtained in 6 ways:

$$9: \quad 1+2+6,\ 1+3+5,\ 1+4+4,\ 2+2+5,\ 2+3+4,\ 3+3+3.$$

$$10: \quad 1+3+6,\ 1+4+5,\ 2+4+4,\ 2+3+5,\ 2+2+6,\ 3+3+4.$$

(a) Compute the probability of the event $1+2+6$ (one point on one die, two points on the other, and 6 points on the remaining die).

(b) Compute the probability of the event $2+4+4$ (two points on one die, and four points on each of the remaining dice).

(c) Find the probabilities of the events $A =$ "A total of 9 points on three dice", and $B =$ "A total of 10 points on three dice".

Solution:

(a) There are 6^3 possible outcomes after rolling the dices. Now, let's find the number of good outcomes:
Since all numbers on dice must be distinct, there are 3! good outcomes (we calculate the number of permutations of $\{1, 2, 4\}$), so the probability is $\frac{6}{6^3} = \boxed{\frac{1}{36}}$.

(b) There are only 3 possible permutations of $\{2, 4, 4\}$, so the probability is $\frac{3}{6^3} = \boxed{\frac{1}{72}}$.

(c) The probability to get three different fixed numbers after rolling three dices is $\frac{1}{36}$, the probability to get two different fixed numbers is $\frac{1}{72}$, and the probability to get all fixed equal numbers is $\frac{1}{6^3}$, by looking at all possible combinations to get 9, and summing, we get that the probability $\mathsf{P}(A) = 3 \cdot \frac{1}{36} + 2 \cdot \frac{1}{72} + \frac{1}{216} = \boxed{\frac{25}{216}}$.
Using the same reasoning, we get $\mathsf{P}(B) = 3 \cdot \frac{1}{36} + 3 \cdot \frac{1}{72} + 0 \cdot \frac{1}{216} = \boxed{\frac{27}{216}}$.

Problem 15

Let A and B be events with probabilities $\mathsf{P}(A) = \frac{3}{4}$ and $\mathsf{P}(B) = \frac{1}{3}$. Show that

$$\frac{1}{12} \leq \mathsf{P}(A \cap B) \leq \frac{1}{3},$$

and give examples to show that both extremes are possible.
Find corresponding bounds for $\mathsf{P}(A \cup B)$.

Solution:

We have (using the fact that P is a non-decreasing set function) that

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cup B) \geq \mathsf{P}(A) + \mathsf{P}(B) - 1 = \frac{1}{12}.$$

Also, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$:

$$\mathsf{P}(A \cap B) \leq \min\{\mathsf{P}(A), \mathsf{P}(B)\} = \frac{1}{3}.$$

These bounds are attained in the following example.
Pick a number at random from $\{1, 2, \ldots, 12\}$.
Taking $A = \{1, 2, \ldots, 9\}$ and $B = \{9, 10, 11, 12\}$, we find that $A \cap B = \{9\}$, and so:

$$\mathsf{P}(A) = \frac{3}{4}, \quad \mathsf{P}(B) = \frac{1}{3}, \quad \mathsf{P}(A \cap B) = \frac{1}{12}.$$

To attain the upper bound for $\mathsf{P}(A \cap B)$, take $A = \{1, 2, \ldots, 9\}$ and $B = \{1, 2, 3, 4\}$.
Likewise we have in this case

$$\mathsf{P}(A \cup B) \leq \min\{\mathsf{P}(A) + \mathsf{P}(B), 1\} = \boxed{1},$$
$$\mathsf{P}(A \cup B) \geq \max\{\mathsf{P}(A), \mathsf{P}(B)\} = \boxed{\frac{3}{4}}.$$

These bounds are attained in the examples above.