Quiz

Suppose that X is a random observation from a uniform distribution on the interval $(0, \delta)$, where $\delta > 1$ and that one wants to estimate $\theta = \mathsf{P}(X > 1) = 1 - 1/\delta$. Consider the following estimator T of θ:

$$T = \begin{cases} 1, & X > 1, \\ 0, & X \leq 1. \end{cases}$$

1. Is T an unbiased estimator?
2. Find the mean squared error of the estimator.

Solution:

1. An estimator T is unbiased if:
$$\mathsf{E}(T) = \theta.$$
Let's find $\mathsf{E}(T)$:
$$\mathsf{E}(T) = 1 \cdot \mathsf{P}(X > 1) + 0 \cdot \mathsf{P}(X \leq 1) = 1 \cdot \theta + 0 = \theta.$$
An estimator T $\boxed{\text{is unbiased}}$.
2. The MSE of T:
$$\text{MSE}(T) = \mathsf{V}(T) + \text{Bias}^2(T).$$
The variance of T:
$$\mathsf{V}(T) = \mathsf{E}\left(T^2\right) - \mathsf{E}(T)^2 = \left[1^2 \cdot \mathsf{P}(X > 1) - 0^2 \cdot \mathsf{P}(X \leq 1)\right] - \theta^2 = \theta - \theta^2.$$
Since $\text{Bias}(T) = 0$:
$$\text{MSE}(T) = \boxed{\theta - \theta^2}.$$

Problem 1

The coin was tossed 10 times, and 8 heads were observed. Can the coin be considered fair? (Use significance level $\alpha = 0.05$.)

Solution:

We have a sample of Bernoulli trials $\{X_1, \ldots, X_{10}\}$:

$$X_i \sim \text{Bernoulli}(p),$$

where p is a probability to get heads after one toss.
The null hypothesis:

$$\mathrm{H}_0 : p = \frac{1}{2}.$$

There is a suspicion that the probability of heads is greater than $\frac{1}{2}$, so we will test the null hypothesis versus right-sided alternative, so we would be able to reject H_0 more easily:

$$\mathrm{H}_1 : p > \frac{1}{2}.$$

The test statistic is chosen to be:

$$T = X_1 + \ldots + X_{10},$$

which indicates with its value how probable hypothesis H_0 is – greater value means lesser probability.
T has a binomial distribution:

$$T \sim \text{Bin}(10, p),$$

so we can calculate p-value for an outcome $T(x_1, \ldots, x_{10}) = t_0 = 8$:

$$p\text{-val} = \mathsf{P}(T \geq 8)\Big|_{\mathrm{H}_0} = C_{10}^{8}\left(\frac{1}{2}\right)^{10} + C_{10}^{9}\left(\frac{1}{2}\right)^{10} + C_{10}^{10}\left(\frac{1}{2}\right)^{10} = \frac{56}{1024} \approx 0.055.$$

p-value is greater than the chosen Type I error (or significance level) $\alpha = 0.05$, which means that the null hypothesis H_0 is $\boxed{\text{not rejected}}$ for such level of significance.
In other words, there is no moderately significant evidence that the probability of heads is greater than $\frac{1}{2}$.
If the result of an experiment would have been 9:

$$p\text{-val} = \mathsf{P}(T \geq 9)\Big|_{\mathrm{H}_0} = C_{10}^{9}\left(\frac{1}{2}\right)^{10} + C_{10}^{10}\left(\frac{1}{2}\right)^{10} = \frac{11}{1024} \approx 0.011,$$

which is smaller than chosen α. It would mean that H_0 is rejected and there is a moderately significant evidence that the probability of heads is greater than $\frac{1}{2}$.

Problem 2

Junior researcher Angela is presenting her half-year project about speed of blood clotting in front of the Head of her laboratory. That speed is normally distributed with population standard deviation $\sigma = 3$ minutes. Angela is very nervous and in the very responsible moment she has forgotten the resultant value of true population speed μ – it's either 6 or 9 minutes. Presentation slide claims that they had 16 observations and the sample mean is 7 minutes, so Angela assumes that the correct value is 6.

1. Find the critical value $\overline{x}_{\text{crit}}$ of sample mean for a hypothesis $\text{H}_0 : \mu = 6$, which would guarantee that it's true within 95% confidence level.
2. Find the p-value for the hypothesis H_0.
3. Preserving confidence level from part 1, find the probability of Type II error, using aforementioned alternative $\text{H}_1 : \mu = 9$.

Solution:

1. Since we have following hypotheses considered:

$$\text{H}_0 : \mu = 6 \quad \text{vs} \quad \text{H}_1 : \mu = 9,$$

the critical region is to the right of H_0. (Note: in this regard the alternative $\text{H}_1 : \mu = 9$ is equivalent to $\text{H}_1 : \mu > 9$.)

The test statistic is sample mean $\overline{X}$, and the rejection rule is $\overline{X} \geq \overline{x}_{\text{crit}}$.

The significance level is $\alpha = 1 - 0.95 = 0.05$, which we can use to find $\overline{x}_{\text{crit}}$ from definition of type I error (reject H_0 incorrectly):

$$\alpha = \mathsf{P}(\text{H}_1 \mid \text{H}_0) = \mathsf{P}\left(\overline{X} \geq \overline{x}_{\text{crit}} \mid \mu = 6\right).$$

$\overline{X}$ is distributed as follows:

$$\overline{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) = \mathcal{N}\left(\mu, \frac{9}{16}\right),$$

thus

$$\alpha = \mathsf{P}\left(Z \geq \frac{\overline{x}_{\text{crit}} - 6}{3/4}\right) = 0.05,$$

$$\frac{\overline{x}_{\text{crit}} - 6}{3/4} = z_{0.05} = 1.645,$$

$$\overline{x}_{\text{crit}} = 1.645 \cdot \frac{3}{4} + 6 \approx \boxed{7.23}.$$

Since $\overline{x} = 7$ is in the critical region ($\overline{x}_{\text{crit}} > \overline{x}$), H_0 is not rejected, so there is no moderately significant evidence that $\mu = 9$ minutes (that μ is greater than 6 minutes).

2. The critical region is to the right, so:

$$p\text{-val} = \mathsf{P}\left(\overline{X} \geq \overline{x} \mid \mu = 6\right) = \mathsf{P}\left(Z \geq \frac{7-6}{3/4}\right) = 1 - \Phi\left(\frac{4}{3}\right) \approx \boxed{0.091}.$$

Since p-val $> \alpha$, H_0 is not rejected, as we have found that out in part 1.

3. By definition of type II error (do not reject H_0 incorrectly):

$$\begin{aligned}\beta &= \mathsf{P}(\mathrm{H}_0 \mid \mathrm{H}_1) = \mathsf{P}\left(\overline{X} < \overline{x}_{\text{crit}} \mid \mu = 9\right) = \\ &= \mathsf{P}\left(Z < \frac{7.23-9}{3/4}\right) \approx \Phi(-2.36) = 1 - \Phi(2.36) \approx \boxed{0.009}.\end{aligned}$$

Problem 3

A firm manufacturing memory chips found that if everything was going right, 10% of them were defective. If the production process was in trouble then 40% were defective. Firm's quality control office tests four memory chips each hour. If two or more of four were defective, production would be shut down to look for trouble.

1. What is Type I and Type II errors here?
2. What is probability that production will be shut down if everything was going right?
3. What is the probability of the missed alarm?
4. Suppose, you have additional information:
   (A) Production goes "out of control" about 10% of hours.
   (B) Testing of one chip costs \$10.
   (C) Missed alarm costs \$10000.
   (D) False alarm costs \$2000.

   Calculate the expected total cost associated with faulty production. Is it better to use another decision rule for detecting the trouble (shut the production if at least one of four tested chips is defective)?
5. New manager suggested testing 100 chips and shutting down production if more than 25 were defective. Calculate expected total cost associated with faulty production for this (part 3) decision rule. Compare with the first two.

Solution:

Let's articulate hypotheses:

$$\begin{aligned} \mathrm{H}_0 &: \quad \text{production is NOT in trouble } (p = 0.1), \\ \mathrm{H}_1 &: \quad \text{production is in trouble } (p = 0.4), \end{aligned}$$

where p is a probability to find a defective memory chip.

1. Type I error: production is considered to in trouble, but actually not.

   Type II error: production is considered to be normal, but actually is in trouble.

2. We are asked to calculate a probability of Type I error α.

   Let $X \sim \text{Bin}(4, p)$ be a number of defective chips in quality control test. The decision rule is $X \geq 2$. The probability α in this case:

$$\begin{aligned}\alpha_1 &= \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_0) = \mathsf{P}(X \geq 2 \mid p = 0.1) = \\ &= 1 - \mathsf{P}(X = 0 \mid p = 0.1) - \mathsf{P}(X = 1 \mid p = 0.1) = \\ &= 1 - C_4^0 \cdot 0.1^0 \cdot 0.9^4 - C_4^1 \cdot 0.1^1 \cdot 0.9^3 = \boxed{0.0523}.\end{aligned}$$

3. We are asked to calculate a probability of Type II error β:

$$\begin{aligned}\beta_1 &= \mathsf{P}(\mathrm{H}_0 \mid \mathrm{H}_1) = \mathsf{P}(X < 2 \mid p = 0.4) = \\ &= \mathsf{P}(X = 0 \mid p = 0.4) + \mathsf{P}(X = 1 \mid p = 0.4) = \\ &= C_4^0 \cdot 0.4^0 \cdot 0.6^4 + C_4^1 \cdot 0.4^1 \cdot 0.6^3 = \boxed{0.4752}.\end{aligned}$$

4. The expected total cost of a quality control test:

$$\begin{aligned}\text{ETC}_1 = \#\ \text{chips} \cdot \$(\text{1 chip test}) &+ \\ + \mathsf{P}(\text{malfunc}) \cdot \mathsf{P}(\text{missed alarm}) \cdot \$(\text{missed alarm}) &+ \\ + \mathsf{P}(\text{func}) \cdot \mathsf{P}(\text{false alarm}) \cdot \$(\text{false alarm}) &= \\ = 4 \cdot \$10 + 0.1 \cdot 0.4752 \cdot \$10000 + 0.9 \cdot 0.0523 \cdot \$2000 &= \boxed{\$609.34}.\end{aligned}$$

   Now let's use another decision rule $X \geq 1$ and calculate new ETC.

   The Type I error probability:

$$\begin{aligned}\alpha_2 &= \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_0) = \mathsf{P}(X \geq 1 \mid p = 0.1) = \\ &= 1 - \mathsf{P}(X = 0 \mid p = 0.1) = 1 - C_4^0 \cdot 0.1^0 \cdot 0.9^4 = 0.3439.\end{aligned}$$

   The Type II error probability:

$$\begin{aligned}\beta_2 &= \mathsf{P}(\mathrm{H}_0 \mid \mathrm{H}_1) = \mathsf{P}(X < 1 \mid p = 0.4) = \\ &= \mathsf{P}(X = 0 \mid p = 0.4) = C_4^0 \cdot 0.4^0 \cdot 0.6^4 = 0.1296.\end{aligned}$$

   The ETC:

$$\text{ETC}_2 = 4 \cdot \$10 + 0.1 \cdot 0.1296 \cdot \$10000 + 0.9 \cdot 0.3439 \cdot \$2000 = \$788.62.$$

   Since $\text{ETC}_2 > \text{ETC}_1$, it is better to use $\boxed{\text{the first}}$ decision rule.

5. The new variable for quality control is $X \sim \text{Bin}(100, p)$ with decision rule $X > 25$.

   The Type I error probability:

$$\alpha_3 = \mathsf{P}(\text{H}_1 \mid \text{H}_0) = \mathsf{P}(X > 25 \mid p = 0.1).$$

   According to the De Moivre–Laplace theorem:

$$X \sim \text{Bin}\,(100, p) \overset{d}{\approx} X_{\text{CLT}} \sim \mathcal{N}\,(100p, 100p(1-p))\,,$$

   and since we approximate discrete distribution with continuous, we should conduct continuity correction. Point $X = 25$ is excluded:

$$\alpha_3 \approx \mathsf{P}(X_{\text{CLT}} > 25.5 \mid p = 0.1) = \mathsf{P}\left(Z > \frac{25.5 - 100 \cdot 0.1}{\sqrt{100 \cdot 0.1 \cdot 0.9}}\right) = 1 - \Phi(5.17) \approx 10^{-7}.$$

   The Type II error probability:

$$\begin{aligned}\beta_3 = \mathsf{P}(\text{H}_1 \mid \text{H}_0) = \mathsf{P}(X \leq 25 \mid p = 0.4) \approx \mathsf{P}(X_{\text{CLT}} \leq 25.5 \mid p = 0.4) = \\ = \mathsf{P}\left(Z \leq \frac{25.5 - 100 \cdot 0.4}{\sqrt{100 \cdot 0.4 \cdot 0.6}}\right) \approx \Phi(-2.96) = 1 - \Phi(2.96) = 0.0015.\end{aligned}$$

   The ETC:

$$\text{ETC}_3 = 100 \cdot \$10 + 0.1 \cdot 0.0015 \cdot \$10000 + 0.9 \cdot 10^{-7} \cdot \$2000 \approx \boxed{\$1001.5}.$$

   Results for ETC are following:

$$\text{ETC}_3 > \text{ETC}_2 > \text{ETC}_1.$$

   It means that the third decision rule is the worst one.

Problem 4

You have a coin with $\mathsf{P}(\text{tail}) = p$. You test the null hypothesis that the coin is a fair one. You flip the coin 5 times and if number of tails is 2 or 3, you do not reject the null hypothesis, otherwise you suppose the coin is biased.

1. What is significance level of the test?
2. Plot the OCC function and the power function of the test.
3. Find values of the power function at the $p = 0.3$ and $p = 0.7$.

Solution:

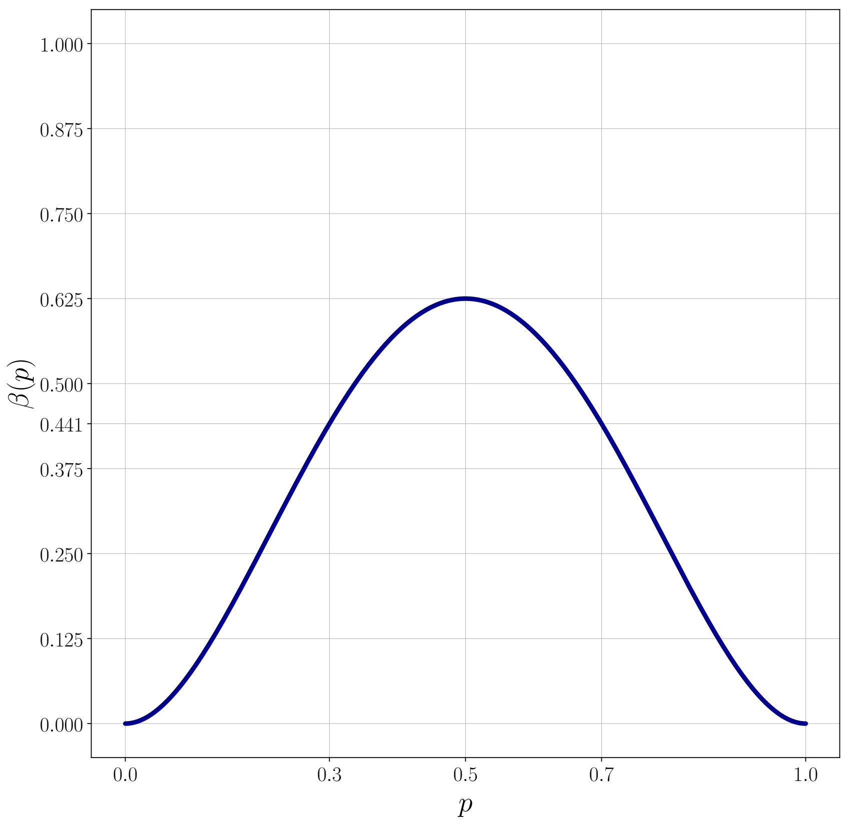


Figure 1: OCC of the test.

Let $X \sim \text{Bin}(4, p)$ be a number of tails after 5 tosses.

Hypotheses in consideration:

$$\text{H}_0 : p = \frac{1}{2} \quad \text{vs} \quad \text{H}_1 : p \neq \frac{1}{2}.$$

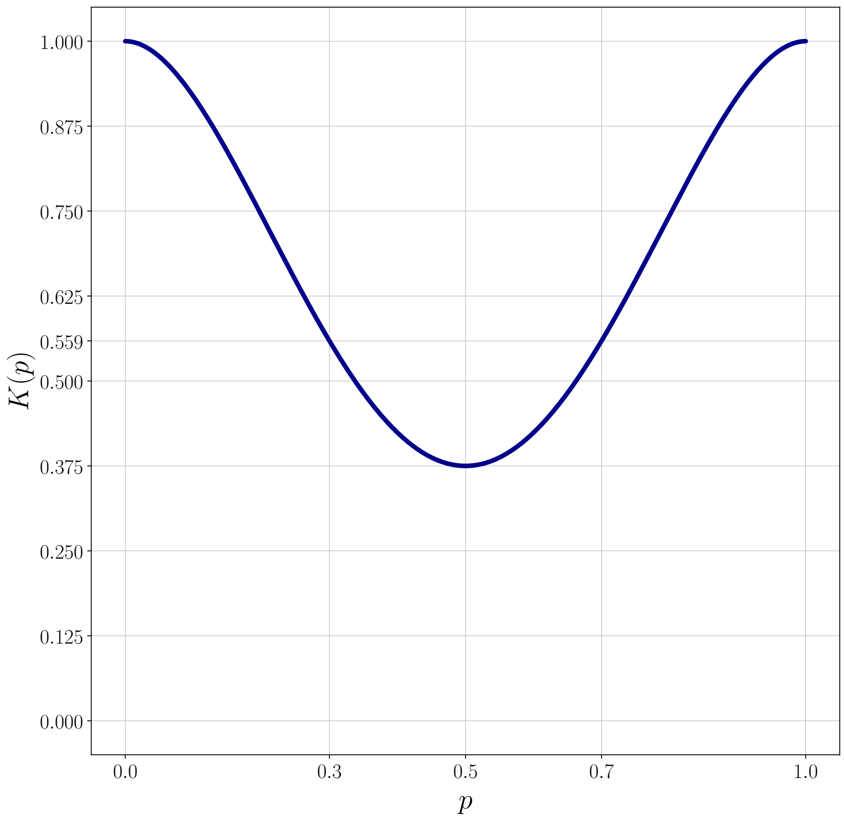


Figure 2: Graph of the power of the test.

The rejection rule is $X = \{0, 1, 4, 5\}$.

1. The probability of Type I error:

$$\alpha = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_0) = \mathsf{P}\left(X = \{0, 1, 4, 5\} \mid p = \frac{1}{2}\right) =$$
$$= 1 - \mathsf{P}\left(X = 2 \mid p = \frac{1}{2}\right) - \mathsf{P}\left(X = 3 \mid p = \frac{1}{2}\right) =$$
$$= 1 - C_5^2 \cdot \left(\frac{1}{2}\right)^5 - C_5^3 \cdot \left(\frac{1}{2}\right)^5 = \boxed{\frac{3}{8}}.$$

2. The probability of Type II error:

$$\beta(p) = \mathsf{P}(\mathrm{H}_0 \mid \mathrm{H}_1) = \mathsf{P}\left(X = \{2, 3\} \mid p \neq \frac{1}{2}\right) =$$
$$= \mathsf{P}\left(X = 2 \mid p \neq \frac{1}{2}\right) + \mathsf{P}\left(X = 3 \mid p \neq \frac{1}{2}\right) =$$
$$= \boxed{C_5^2 \cdot p^2 \cdot (1-p)^3 - C_5^3 \cdot p^3 \cdot (1-p)^2,\ p \neq \frac{1}{2}}.$$

The OCC of $\beta(p)$ is shown in the fig. 1.

The power $K(p)$ is a probability to correctly reject H_0:

$$K(p) = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_1) = 1 - \beta(p) =$$
$$= \boxed{1 - C_5^2 \cdot p^2 \cdot (1-p)^3 - C_5^3 \cdot p^3 \cdot (1-p)^2, \ p \neq \frac{1}{2}}.$$

The graph of $K(p)$ is shown in the fig. 2.

3. Values of $K(0.3)$ and $K(0.7)$ are identical, since values of $K(p)$ are symmetrical with respect to $\frac{1}{2}$:

$$K(0.3) = K(0.7) = 1 - C_5^2 \cdot 0.3^2 \cdot 0.7^3 - C_5^3 \cdot 0.3^3 \cdot 0.7^2 = \boxed{0.559}.$$

Problem 5

Random variable X has normal distribution $\mathcal{N}(\mu, \sigma^2)$. Let σ be equal to 25 and sample size be equal to 100. You test null hypothesis $\mathrm{H}_0 : \mu = 100$ against the alternative hypothesis $\mathrm{H}_1 : \mu < 100$. Significance level of the test is 10%. You reject the null hypothesis if $\overline{X} < c$. Plot the power function of the test.

Solution:

Let's find c first.

The test statistic is a sample mean $\overline{X}$ with following distribution:

$$\overline{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) = \mathcal{N}\left(\mu, \frac{25^2}{100}\right).$$

The critical region is to the left, so the equation for a probability of Type I error:

$$\alpha = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_0) = \mathsf{P}\left(\overline{X} < c \mid \mu = 100\right) = \mathsf{P}\left(Z < \frac{c - 100}{25/10}\right).$$

From problem statement we know that $\alpha = 0.1$. Thus:

$$\frac{c - 100}{25/10} = -z_{0.1} \approx -1.29,$$

$$c = 100 - 1.29 \cdot \frac{25}{10} = 96.775.$$

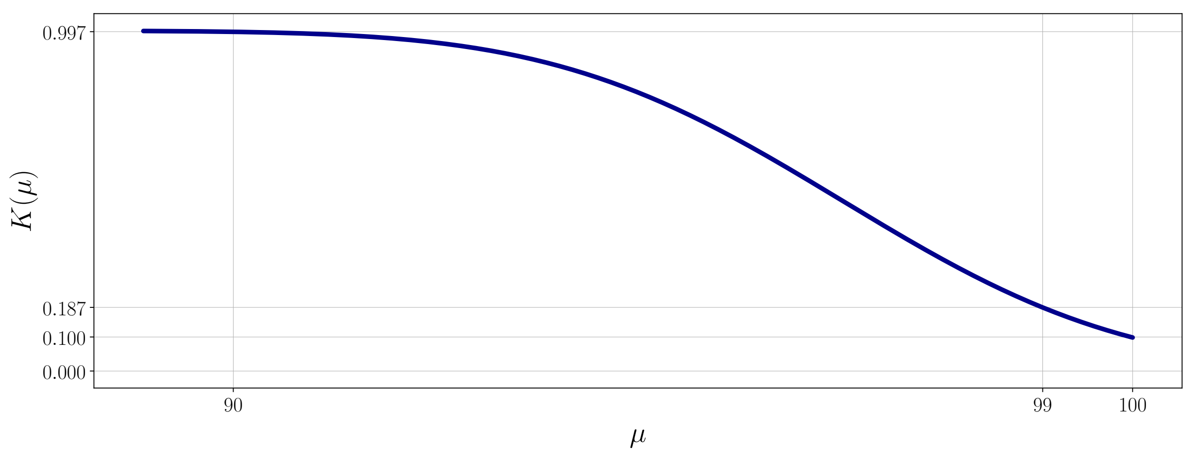


Figure 3: Graph of the power of the test.

The power $K(\mu)$ is a probability to correctly reject H_0:

$$K(\mu) = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_1) = \mathsf{P}\left(\overline{X} < c \mid \mu < 100\right) =$$
$$= \mathsf{P}\left(Z < \frac{96.775 - \mu}{25/10} \middle| \mu < 100\right) = \boxed{\Phi\left(\frac{96.775 - \mu}{2.5}\right),\ \mu < 100}.$$

The graph of $K(\mu)$ is shown in the fig. 3.

Problem 6

Let X be $\mathcal{N}(\mu, 10^2)$. To test $\mathrm{H}_0 : \mu = 80$ against alternative $\mathrm{H}_1 : \mu > 80$ the critical region $\overline{x} > x_c = 83$ was chosen for the sample of size $n = 25$.

1. What is the power function $K(\mu)$ of this test?
2. What is the significance level of this test?
3. What are the values $K(80), K(83), K(86)$?
4. Sketch the graph of the power function and the OCC function.
5. What is the p-value corresponding to $\overline{x} = 83.41$?

Solution:

The test statistic is a sample mean $\overline{X}$ with following distribution:

$$\overline{X} \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) = \mathcal{N}\left(\mu, \frac{10^2}{25}\right).$$

1. The power $K(\mu)$ is a probability to correctly reject H_0:

$$K(\mu) = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_1) = \mathsf{P}\left(\overline{X} > x_c \mid \mu > 80\right) =$$
$$= \mathsf{P}\left(Z > \frac{83-\mu}{10/5} \middle| \mu > 80\right) = \boxed{\Phi\left(\frac{\mu - 83}{2}\right),\ \mu > 80}.$$

2. A significance level is probability of Type I error:

$$\alpha = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_0) = \mathsf{P}\left(\overline{X} > x_c \mid \mu = 80\right) = \mathsf{P}\left(Z > \frac{83-80}{10/5}\right) =$$
$$= 1 - \Phi(1.5) \approx 1 - 0.933 = \boxed{0.067}.$$

3. The value of a power of the test in $\mu = 80$ is a probability of Type I error:

$$K(80) = \alpha = \boxed{0.067},$$

since it's a value of μ in H_0.

The value of a power of the test in $\mu = x_c = 83$ is one half:

$$K(83) = K(x_c) = \boxed{0.5}.$$

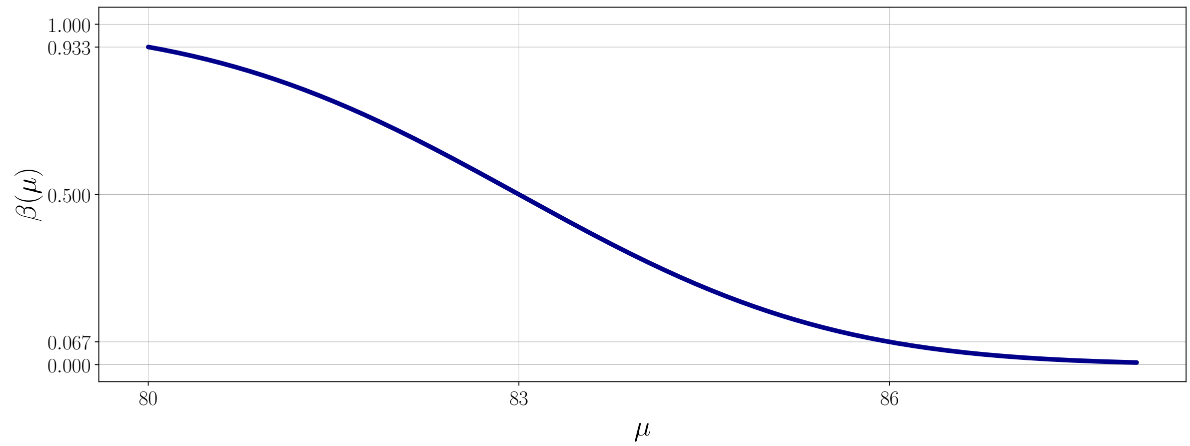


Figure 4: OCC of the test.

The value of a power of the test in $\mu = 86$ is a confidence level:

$$K(86) = 1 - \alpha = 1 - 0.067 = \boxed{0.933},$$

since it's such value of μ that is at the same distance from x_c as value of μ in H_0, but in another direction.

4. The probability of Type II error:

$$\beta(\mu) = \mathsf{P}(\mathrm{H}_0 \mid \mathrm{H}_1) = 1 - K(\mu) =$$
$$= 1 - \Phi\left(\frac{\mu - 83}{2}\right), \ \mu > 80 = \boxed{\Phi\left(\frac{83 - \mu}{2}\right), \ \mu > 80}.$$

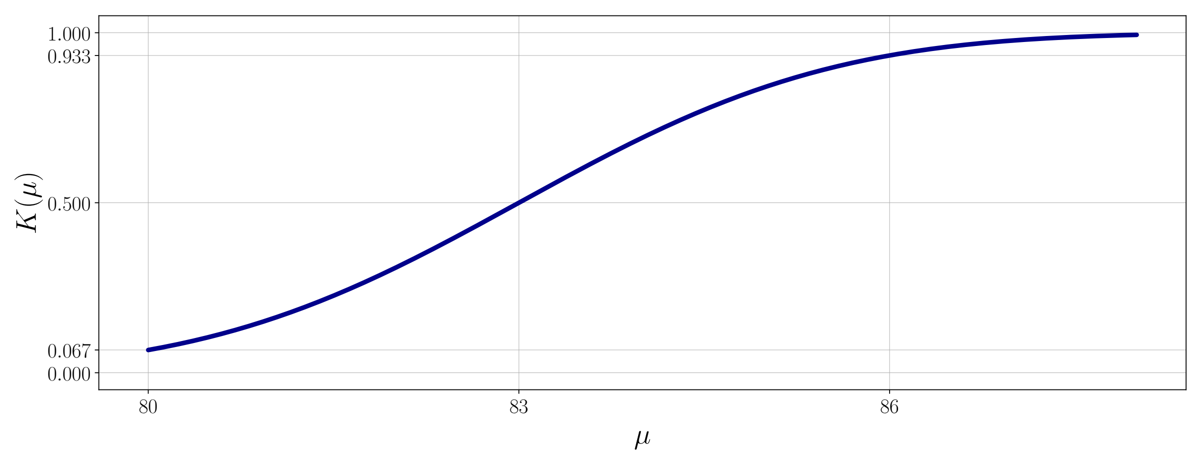


Figure 5: Graph of the power of the test.

The OCC of $\beta(p)$ is shown in the fig. 4 and the graph of $K(\mu)$ is shown in the fig. 5.

5. The critical region is to the right, so:

$$p\text{-val} = \mathsf{P}\left(\overline{X} \geq \overline{x} \mid \mu = 80\right) = \mathsf{P}\left(Z \geq \frac{83.41 - 80}{10/5}\right) \approx 1 - \Phi\left(1.705\right) \approx \boxed{0.044}.$$

Since p-val $\leq \alpha = 0.067$, H_0 is rejected, and there is a significant evidence that $\mu > 80$.

Problem 7

Let X be a Bernoulli random variable with parameter p.

We would like to test the null hypothesis $\mathrm{H}_0 : p \leq 0.4$ against the alternative hypothesis $\mathrm{H}_1 : p > 0.4$.

For the test statistic let's use $Y = \sum\limits_{i=1}^{n} X_i$, where $\{X_1, \ldots, X_n\}$ is a random sample of size n from this Bernoulli distribution.

Let the critical region be of the form $C = \{y : y \geq c\}$.

1. Let $n = 100$. On the same set of axes, sketch the graphs of the power function corresponding to the three critical regions: $C_1 = \{y : y \geq 40\}$, $C_2 = \{y : y \geq 50\}$, $C_3 = \{y : y \geq 60\}$. Use normal approximation to compute the probabilities.
2. Let $C_n = \{y : y \geq 0.45n\}$. On the same set of axes, sketch the graphs of the power function corresponding to the three samples of size 10, 100, and 1000.

Solution:

The power $K(p)$ is a probability to correctly reject H_0:

$$K(p) = \mathsf{P}(\mathrm{H}_1 \mid \mathrm{H}_1) = \mathsf{P}\left(Y \geq c \mid p > 0.4\right).$$

The distribution of test statistic Y:

$$Y \sim \mathrm{Bin}(n, p).$$

According to the De Moivre–Laplace theorem:

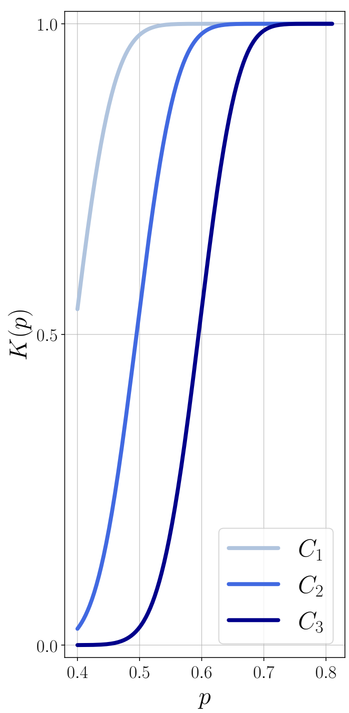
$$Y \sim \mathrm{Bin}\left(n, p\right) \overset{d}{\approx} Y_{\mathrm{CLT}} \sim \mathcal{N}\left(np, np(1-p)\right),$$

and since we approximate discrete distribution with continuous, we should conduct continuity correction. Point $Y = c$ is included:

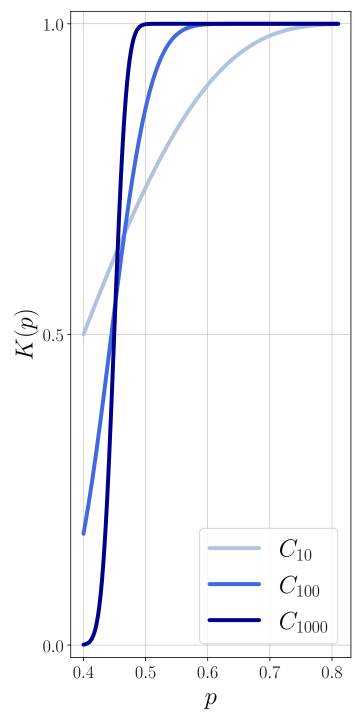
$$K(p) \approx \mathsf{P}(Y_{\mathrm{CLT}} > c - 0.5 \mid p > 0.4) = \mathsf{P}\left(Z > \frac{c - 0.5 - np}{\sqrt{np(1-p)}} \,\middle|\, p > 0.4\right) =$$

$$= 1 - \Phi\left(\frac{c - 0.5 - np}{\sqrt{np(1-p)}}\right), \ p > 0.4 = \boxed{\Phi\left(\frac{np + 0.5 - c}{\sqrt{np(1-p)}}\right), \ p > 0.4}.$$

1. Graphs of power functions for $n = 100$ and $c = \{40, 50, 60\}$ are shown in the fig. 6a.
2. Graphs for $n = \{10, 100, 1000\}$ and $c = 0.45n$ are shown in the fig. 6b.



(a) $n = 100, c = \{40, 50, 60\}$.



(b) $n = \{10, 100, 1000\}, c = 0.45n$.

Figure 6: Graphs of the power of the test.