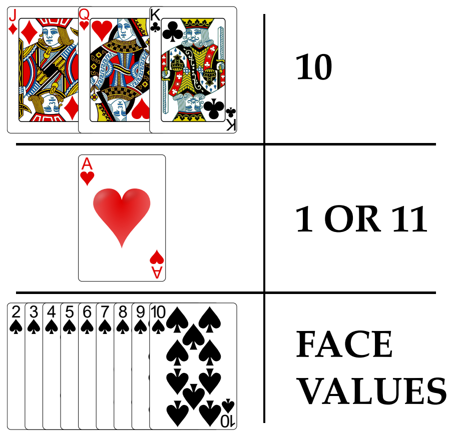
Quiz

Remember BlackJack? You're given 2 random cards from the same deck of 52.



What's the probability of getting:

(a) 21 pts,

(b) 20 pts,

(c) 12 pts,

(d) 10 pts?

Note: ace's value maximizes score if it does not exceed 21.

Solution:

(a) One possible combination: $11 + 10$.

$11 + 10$: $1/4$ aces; $1/16$ tens, jacks, queens and kings.

$$\mathsf{P}(21) = \frac{C_4^1 C_{16}^1}{C_{52}^2} = 2 \cdot \frac{4}{52} \cdot \frac{16}{51} = \boxed{\frac{32}{663} \approx 4.8\%}.$$

(b) Two possible combinations: $10 + 10$ and $11 + 9$.

$10 + 10$: $2/16$ tens, jacks, queens and kings.
$11 + 9$: $1/4$ aces; $1/4$ nines.

$$\mathsf{P}(20) = \frac{C_{16}^2 + C_4^1 C_4^1}{C_{52}^2} = \frac{16}{52} \cdot \frac{15}{51} + 2 \cdot \frac{4}{52} \cdot \frac{4}{51} = \boxed{\frac{4}{39} \approx 10.3\%}.$$

(c) Six possible combinations: $11+1$, $10+2$, $9+3$, $8+4$, $7+5$ and $6+6$.

$11+1$: $2/4$ aces, since $11+11=22>21$.

$10+2$: $1/16$ tens, jacks, queens and kings; $1/4$ deuces.

$9+3$: $1/4$ nines; $1/4$ threes.

$8+4$: $1/4$ eights; $1/4$ fours.

$7+5$: $1/4$ sevens; $1/4$ fives.

$6+6$: $2/4$ sixes.

$$\mathsf{P}(12)=\frac{C_{16}^1C_4^1+3\cdot C_4^1C_4^1+2\cdot C_4^2}{C_{52}^2}=2\cdot\frac{16}{52}\cdot\frac{4}{51}+3\cdot 2\cdot\frac{4}{52}\cdot\frac{4}{51}+2\cdot\frac{4}{52}\cdot\frac{3}{51}=\boxed{\frac{62}{663}\approx 9.4\%}.$$

(d) Four possible combinations: $8+2$, $7+3$, $6+4$ and $5+5$. $9+1$ is impossible, since in that case ace would take value of 11 ($9+11=20<21$).

$8+2$: $1/4$ eights; $1/4$ deuces.

$7+3$: $1/4$ sevens; $1/4$ threes.

$6+4$: $1/4$ sixes; $1/4$ fours.

$5+5$: $2/4$ fives.

$$\mathsf{P}(10)=\frac{3\cdot C_4^1C_4^1+C_4^2}{C_{52}^2}=3\cdot 2\cdot\frac{4}{52}\cdot\frac{4}{51}+\frac{4}{52}\cdot\frac{3}{51}=\boxed{\frac{9}{221}\approx 4.1\%}.$$

Problem 1

Two points X and Y are randomly chosen on an interval $OA = [0, 1]$. Find the probability of each of the following events:

(a) A distance between X and O is less than $\frac{1}{10}$.

(b) A distance between X and O is between 0.7 and 0.705.

(c) A distance between X and O is equal to 0.7

(d) A distance between X and Y is less than 0.5

(e) A distance between X and Y is equal to $\frac{1}{3}$.

(f) Length of XY is less than the distance between O and the closest point to it.

Solution:

(a) Since X is randomly chosen on [0, 1], the probability that $X < \frac{1}{10}$ is $\boxed{\frac{1}{10}}$

(b) Similarly, the probability that $X \in [0.7, 0.705] = 0.705 - 0.7 = \boxed{0.05}$

(c) Since the cardinality of $[0, 1]$ is continuum, and the only good outcome is a single point, the probability is $\boxed{0}$.

(d) We just have to find the area of the the figure, which is defined as:

$$\begin{cases} |X - Y| \leq 0.5 \\ X \geq 0, X \leq 1 \\ Y \geq 0, Y \leq 1 \end{cases}$$

If you draw this figure, you can easily find that its area is equal to $\boxed{\frac{3}{4}}$

(e) Similarly to (c), the "area" of a segment on our grid that contains all good outcomes is 0, so the probability is $\boxed{0}$

(f) Just consider two cases on X and Y:
If $X < Y$, then we have to find the area of the figure, which is defined by the following equations:

$$\begin{cases} X < Y \\ X \geq 0, X \leq 1 \\ Y \geq 0, Y \leq 1 \\ X \leq Y - X \end{cases}$$

Similarly, if $X > Y$, we have:

$$\begin{cases} X > Y \\ X \geq 0, X \leq 1 \\ Y \geq 0, Y \leq 1 \\ Y \leq X - Y \end{cases}$$

By finding the area of the union of these figures, the get the probability is equal to $\boxed{\frac{1}{2}}$

Problem 2

Consider a round shooting target with a radius of R. Someone is shooting at it with bullets of radius B. Find the probability that a hole made by the shot entirely lies within an interior circle with a radius of r. Assume $R > r > B$.

Solution:

Let's assume the all the circles are concentric. Obviously, the the hole made in the shot will lie in the circle of radius $r \Leftrightarrow$ its center lies inside the circle of radius $r - B$. And if we throw shoot, randomly, then the center can be anywhere inside the circle with radius R, so the probability is the ratio of the area of the circles:

$$\mathsf{P}(\text{Hole lies in a inferior circle of radius } r) = \frac{\pi(r-B)^2}{\pi R^2} = \boxed{\frac{(r-B)^2}{R^2}}$$

Problem 3

A stick of length L is broken in two places. The break points are independent of each other and are chosen at random (uniformly) on the stick. What is the probability that a triangle can be formed using these three pieces of stick?

Solution:

Let's assume the break points are X and Y, and $X < Y$. (Since $X < Y$ is symmetric to $X > Y$, we can just calculate the probability with this assumption, and then multiply by 2). Then, the lengths of the sticks are $X,\ Y - X,\ L - Y$. These pieces form a triangle, if none of them is longer than the sum of two others, so, we have

$$\begin{cases} L - Y < Y, \\ Y - X < L - Y + X, \\ X < L - X. \end{cases}$$

The area of this figure is $\frac{1}{8}$, so by multiplying by two, we get the desired probability is $\boxed{\frac{1}{4}}$

Problem 4

There are two children in a family. It is known that at least one of them is a boy. Find the probability that the other child is also a boy.

Solution:

The sample space contains 4 possible outcomes: BB, BG, GB, GG, where B stands for boy, G stands for girl.

Let A be the event that one child is a boy. Then probability of A is $\mathsf{P}(A) = 3/4$, since this event includes 3 outcomes: BB, BG and GB.

Let C be the event than another child is a boy. Probability that one child is a boy and another one is also a boy is $\mathsf{P}(C \cap A) = 1/4$, since it includes only one outcome BB.

We have to find probability $\mathsf{P}(C \mid A)$. By definition of conditional probability:

$$\mathsf{P}(C \mid A) = \frac{\mathsf{P}(C \cap A)}{\mathsf{P}(A)} = \frac{1/4}{3/4} = \boxed{\frac{1}{3}}.$$

Problem 5

Two coins are tossed. Let's denote events A, B and C the following way:

$$A = \{1^{\text{st}} \text{ coin is heads}\},$$
$$B = \{2^{\text{nd}} \text{ coin is heads}\},$$
$$C = \{\text{only one coin is heads}\}.$$

Are these events collectively independent? Are they pairwise independent? Will the situation change if the coin is not fair?

Solution:

A collection of events is called pairwise independent if any two events from this collection are independent of each other. This means that their conditional probabilities are equal to their own probabilities, as shown below:

$$\mathsf{P}(A \mid B) = \mathsf{P}(A), \qquad \mathsf{P}(C \mid A) = \mathsf{P}(C), \qquad \ldots$$

This collection is termed collectively (or mutually) independent if each event is independent of any combination of the other events. In the case of three events, in addition to the previous pairwise independent equations, we have:

$$\mathsf{P}(A \mid B, C) = \mathsf{P}(A), \qquad \mathsf{P}(B \mid A, C) = \mathsf{P}(B), \qquad \mathsf{P}(C \mid A, B) = \mathsf{P}(C).$$

Event A includes outcomes (HT, HH), event B includes (TH, HH), and event C includes (HT, TH), where H stands for heads and T stands for tails.

(a) The case of fair coin.

Own probabilities of events are all equal to $1/2$, since all three events take 2 outcomes from a sample space of 4 equiprobable outcomes:

$$\mathsf{P}(A) = \mathsf{P}(B) = \mathsf{P}(C) = \frac{1}{2}.$$

Probabilities of pairwise intersections are all equal to $1/4$, since each of them take 1 outcome out of sample space:

$$\mathsf{P}(A \cap B) = \mathsf{P}(A \cap C) = \mathsf{P}(B \cap C) = \frac{1}{4}.$$

Let's calculate one example to check the pairwise independence:

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} = \frac{1/4}{1/2} = \frac{1}{2} = \mathsf{P}(A).$$

It means that events A and B are independent. But since all probabilities of intersections and own probabilities are same in value (1/4 and 1/2 respectively), all conditional probabilities are equal to 1/2:

$$\mathsf{P}(A \mid B) = \mathsf{P}(A \mid C) = \ldots = \mathsf{P}(C \mid B) = \frac{1/4}{1/2} = \frac{1}{2}.$$

Values of conditional probabilities are same with own ones, which means that events A, B and C are $\boxed{\text{pairwise independent}}$.

Consider the probability $\mathsf{P}(A \cap B \cap C)$. The intersection of this events is $\varnothing$, that's why its probability is 0. Let's calculate conditional probability, given combination:

$$\mathsf{P}(A \mid B, C) = \frac{\mathsf{P}(A, B, C)}{\mathsf{P}(B, C)} = \frac{\mathsf{P}(A \cap B \cap C)}{\mathsf{P}(B \cap C)} = \frac{0}{1/4} = 0 \neq \mathsf{P}(A).$$

Since all conditional probabilities, given combinations, include probability of intersection $\mathsf{P}(A \cap B \cap C) = 0$, all of them will be equal to 0. It means that they do not coincide with own probabilities of events, which are equal to 1/2. Thus, events A, B and C are $\boxed{\text{NOT collectively independent}}$.

(b) The case of unfair coin.

Events A, B and C now have different probabilities. Let $\mathsf{P}(H) = p$, then:

$$\mathsf{P}(A) = \mathsf{P}(B) = p(1-p) + p^2 = p, \qquad \mathsf{P}(C) = 2p(1-p).$$

Probabilities od intersections:

$$\mathsf{P}(A \cap B) = p^2, \qquad \mathsf{P}(A \cap C) = \mathsf{P}(B \cap C) = p(1-p).$$

Let's calculate pairwise conditional probabilities:

$$\mathsf{P}(A \mid B) = \mathsf{P}(B \mid A) = \frac{p^2}{p} = p = \mathsf{P}(B) = \mathsf{P}(A).$$

It means that events A and B are independent. But does it mean that the collection of events is pairwise independent? No, let's check conditionals with C:

$$\mathsf{P}(A \mid C) = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2} \neq \mathsf{P}(A).$$

The same goes for $\mathsf{P}(B \mid C)$ and probabilities $\mathsf{P}(C \mid A)$ and $\mathsf{P}(C \mid B)$ (but with value $1-p$). Events A, B and C in this case are $\boxed{\text{NOT pairwise independent}}$.

The probability $\mathsf{P}(A \cap B \cap C)$ even with unfair coin is still 0. It means that events here are also $\boxed{\text{NOT collectively independent}}$.

Problem 6

There are three cards.

- The letter $\mathbb{A}$ is written on both sides of the 1$^{\text{st}}$ card.
- The letter $\mathbb{A}$ is written on both sides of the 2$^{\text{nd}}$ card.
- Letters $\mathbb{A}$ and $\mathbb{B}$ are written on different sides of the 3$^{\text{rd}}$ card.

A random card has been put on the table in such a way that the letter $\mathbb{A}$ is visible. What is the probability that the letter $\mathbb{A}$ is written on the other side of the card?

Solution:

i. Let X be event that A is on the 1$^{\text{st}}$ side of the chosen card, and let Y be event that A is on the 2$^{\text{nd}}$ side of the card.

   Probability that card contains two A-s is:

$$\mathsf{P}(Y \cap X) = \frac{2}{3},$$

   since there 2 such cards out of 3 possible.

   Probability that A is on the first side is:

$$\mathsf{P}(X) = \frac{5}{6},$$

   since there are five A-s and one B.

   We need to find $\mathsf{P}(Y \mid X)$. Using definition of conditional probability:

$$\mathsf{P}(Y \mid X) = \frac{\mathsf{P}(Y \cap X)}{\mathsf{P}(X)} = \frac{2/3}{5/6} = \boxed{\frac{4}{5}}.$$

ii. Let A be event that A is on the 1$^{\text{st}}$ side of the chosen card, and let 1, 2 and 3 be events that the card is of this number. Then from problem statement:

$$\mathsf{P}(1) = \mathsf{P}(2) = \mathsf{P}(3) = \frac{1}{3}, \qquad \mathsf{P}(A \mid 1) = \mathsf{P}(A \mid 2) = 1, \qquad \mathsf{P}(A \mid 3) = \frac{1}{2}.$$

   Probability of A is the same with $\mathsf{P}(X)$ from solution i. and is equal to $5/6$.

   We need to find $\mathsf{P}(1 \cup 2 \mid A)$. Using the total probability and Bayes theorem:

$$\mathsf{P}(1 \cup 2 \mid A) = 1 - \mathsf{P}(3 \mid A) = 1 - \frac{\mathsf{P}(A \mid 3) \cdot \mathsf{P}(3)}{\mathsf{P}(A)} = 1 - \frac{1/2 \cdot 1/3}{5/6} = 1 - \frac{1}{5} = \boxed{\frac{4}{5}}.$$

Problem 7

A system consists of two parallel elements and is operational if at least one of them is working. At a random time, the 1st element is out of order with a probability of 0.1, and the 2nd element is out of order with a probability of 0.2. Someone has informed us that the system is currently operational. What is the probability that the 2nd element is out of order?

Solution:

Let $F1$ be event of the 1st element fail, $F2$ – event of the 2nd element fail, W – system works. From problem statement we know that:

$$\mathsf{P}(F1) = 0.1, \qquad \mathsf{P}(F2) = 0.2.$$

We need to find conditional probability $\mathsf{P}(F2 \mid W)$. Using Bayes theorem:

$$\mathsf{P}(F2 \mid W) = \frac{\mathsf{P}(F2) \cdot \mathsf{P}(W \mid F2)}{\mathsf{P}(W)}.$$

Firstly, let's calculate $\mathsf{P}(W)$. The system works in case if none of elements failed:

$$\mathsf{P}(W) = 1 - \mathsf{P}(F1) \cdot \mathsf{P}(F2) = 1 - \frac{1}{10} \cdot \frac{2}{10} = \frac{49}{50}.$$

After that, let's calculate $\mathsf{P}(W \mid F2)$. The system works, given that the 2nd elements failed, only if the 1st element works:

$$\mathsf{P}(W \mid F2) = 1 - \mathsf{P}(F1) = 1 - \frac{1}{10} = \frac{9}{10}.$$

Thus, the desired probability is:

$$\mathsf{P}(F2 \mid W) = \frac{2/10 \cdot 9/10}{49/50} = \boxed{\frac{9}{49}}.$$