Quiz

For each one of the statements below say whether the statement is true or false explaining your answer. A and B are events such that $0 < \mathsf{P}(A) < 1$ and $0 < \mathsf{P}(B) < 1$.

(a) If A and B are independent, then $\mathsf{P}(A) + \mathsf{P}(B) > \mathsf{P}(A \cup B)$.

(b) If A and B are independent, then $\mathsf{P}(A) + \mathsf{P}(B) \leq 1$.

(c) If $\mathsf{P}(A) < \mathsf{P}(B)$, then $A \subset B$.

(d) If $\mathsf{P}(A \mid B) = \mathsf{P}(B \mid A)$ and $\mathsf{P}(A \cap B) > 0$, then $\mathsf{P}(A) = \mathsf{P}(B)$.

Solution:

(a) True.

$$\begin{aligned}\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A \cap B) &= \mathsf{P}(A) + \mathsf{P}(B) - \mathsf{P}(A)\mathsf{P}(B)\\ &< \mathsf{P}(A) + \mathsf{P}(B),\end{aligned}$$

since $0 < \mathsf{P}(A) < 1$ and $0 < \mathsf{P}(B) < 1$, hence $\mathsf{P}(A)\mathsf{P}(B) > 0$.

(b) False.

Consider two different dice and let A be the event that the first die is not showing 6 and B be the event that the second die is not showing 6 . The two events are independent but:

$$\mathsf{P}(A) + \mathsf{P}(B) = \frac{5}{6} + \frac{5}{6} = \frac{10}{6} > 1.$$

(c) False.

Consider a fair die and let $A = \{1, 2\}$ and $B = \{2, 3, 4\}$. Clearly $\mathsf{P}(A) < \mathsf{P}(B)$, but $A \not\subset B$.

(d) True.

Note that:

$$\mathsf{P}(A \mid B) = \mathsf{P}(B \mid A)$$

implies:

$$\frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(A)}$$

and since $\mathsf{P}(A \cap B) > 0$ we conclude that:

$$\mathsf{P}(A) = \mathsf{P}(B).$$

Problem 1

Consider a high-risk population where 5% of people have COVID-19. A diagnostic test is correct in 95% of cases if a person has COVID-19 and in 90% of cases if a person does not have COVID-19. If a person tests positive (indicating COVID-19), what is the probability that the person does not have COVID-19?

Solution:

Let's denote event of actually having a disease as D, and event of test indicating disease as I. Then probabilities in problem statement are following:

$$\mathsf{P}(D) = 5\%, \qquad \mathsf{P}(I \mid D) = 95\%, \qquad \mathsf{P}\left(\overline{I} \mid \overline{D}\right) = 90\%.$$

Since events $\left(D, \overline{D}\right)$ and $\left(I, \overline{I}\right)$ are mutually exclusive:

$$\mathsf{P}\left(I \mid \overline{D}\right) = 1 - \mathsf{P}\left(\overline{I} \mid \overline{D}\right) = 10\%, \qquad \mathsf{P}\left(\overline{D}\right) = 1 - \mathsf{P}(D) = 95\%.$$

We need to find probability $\mathsf{P}\left(\overline{D} \mid I\right)$. Since we have only conditional probabilities given D and $\overline{D}$, we will use Bayes theorem to swap events:

$$\mathsf{P}\left(\overline{D} \mid I\right) = \mathsf{P}\left(I \mid \overline{D}\right) \cdot \frac{\mathsf{P}\left(\overline{D}\right)}{\mathsf{P}\left(I\right)}.$$

The only probability we don't know in the previous equation is $\mathsf{P}\left(I\right)$. Let's find it with total probability:

$$\mathsf{P}\left(I\right) = \mathsf{P}(I \mid D) \cdot \mathsf{P}(D) + \mathsf{P}\left(I \mid \overline{D}\right) \cdot \mathsf{P}\left(\overline{D}\right).$$

Substituting everything into the $\mathsf{P}\left(\overline{D} \mid I\right)$:

$$\mathsf{P}\left(\overline{D} \mid I\right) = \frac{10\% \cdot 95\%}{95\% \cdot 5\% + 10\% \cdot 95\%} = \boxed{\frac{2}{3}}.$$

Problem 2

You have a biased coin for which $\mathsf{P}(H) = p$. You toss the coin 20 times. What is the probability that:

(a) You observe first 8 heads and then 12 tails.

(b) You observe 8 heads and 12 tails.

(c) You observe more than 8 heads and more than 8 tails.

Solution:

Let p be the probability that you observe a head. Then $1 - p$ is the probability that you observe a tail.

(a) Since there is only one possible order: $\boxed{p^8\,(1-p)^{12}}$.

(b) Here is C_{20}^{8} possible combinations of heads and tails: $\boxed{C_{20}^{8}\,p^8\,(1-p)^{12}}$.

(c) Combinations of (heads, tails), that satisfy the requirement, are (9,11), (10,10) and (11,9): $\boxed{C_{20}^{9}\,p^9\,(1-p)^{11} + C_{20}^{10}\,p^{10}\,(1-p)^{10} + C_{20}^{11}\,p^{11}\,(1-p)^{9}}$.

Problem 3

Suppose a good password must consist of two lowercase letters (a to z), followed by one capital letter (A to Z), followed by four digits. For example, "*ejT3018*" is a good password.

(a) Find the total number of good passwords.

(b) A hacker wrote a program that randomly generated 108 good passwords (one password could be generated more than once). What is the probability that at least one of the generated passwords matches the password of a particular user?

(c) Answer the question 2 assuming that the program generated 108 distinct passwords.

Solution:

(a) There are 26 letters (lowercase and capital) and 10 digits. Therefore there are $\boxed{N = 26^3 \cdot 10^4}$ possible combinations of good passwords, since characters can coincide.

(b) Probability that at least one password matches password of a particular user is equivalent to negation of probability that none of the passwords match the user's one: $\mathsf{P}(> 1) = 1 - \mathsf{P}(0)$. Probability of k coincidences out of 108 trials is given by binomial distribution: $\mathsf{P}(k) = C_{108}^k \left(\frac{1}{N}\right)^k \left(1 - \frac{1}{N}\right)^{108-k}$. Therefore $1 - \mathsf{P}(0) = 1 - \underbrace{C_{108}^0}_{1} \cdot \underbrace{\left(\frac{1}{N}\right)^0}_{1} \left(1 - \frac{1}{N}\right)^{108} =$

$\boxed{1 - \left(1 - \frac{1}{26^3 \cdot 10^4}\right)^{108}}$.

(c) The difference here is that 108 passwords are taken without replacement. It means that probability of none coinciding passwords is given by choosing 108 passwords out of $N - 1$ incorrect ones, divided by cardinality of sample set: $1 - \mathsf{P}(0) = 1 - \frac{C_{N-1}^{108}}{C_N^{108}} =$

$1 - \frac{(N-108)!}{N!} \cdot \frac{(N-1)!}{(N-109)!} = 1 - \frac{N-108}{N} = \boxed{\frac{108}{26^3 \cdot 10^4}}$.

The difference between two probabilities is in the 13^{th} decimal: $\mathsf{P}(c) - \mathsf{P}(b) \approx 1,87 \cdot 10^{-13}$. It means that the probability to find at least one correct password out of $26^3 \cdot 10^4$ in 108 trials is a bit greater in case when passwords are distinct, which makes perfect sense.

Problem 4

The student has learned 20 out of 25 exam questions before the exam. She will be asked 3 different questions. If she answers all the questions, she will receive an excellent mark; if she answers 2 questions, she will receive a good mark; and if she answers 0 or 1 question, she will receive an unsatisfactory mark. Find the probability of the following:

(a) obtaining an excellent mark,

(b) receiving an unsatisfactory mark,

(c) passing the exam,

(d) passing the exam if she knows the answer to the first question,

(e) knowing two out of three questions, given that she passed the exam.

Solution:

(a) Let's calculate the number of good outcomes. Because the student has to get an excellent mark, she has to know all the questions, and she has learned 20 of 25 exam questions, so, there are 20 ways to choose the first questions, 19 ways to choose the second after that (because all the 3 questions are different) and 18 ways to choose the third. Now let's find the number of all possible outcomes: there are, obviously, 25 ways to choose any first question, 24 ways for the second, and 23 for the third, so the desired probability is $\boxed{\dfrac{20 \cdot 19 \cdot 18}{25 \cdot 24 \cdot 23}}$.

Also, if the count the outcomes as unordered, and not ordered, triplets, we can get $\dfrac{C_{20}^3}{C_{25}^3}$, which is the same.

(b)

$$\mathsf{P}(\text{Unsatisfactory mark}) = 1 - \mathsf{P}(\text{Excellent mark}) - \mathsf{P}(\text{Good mark}).$$

We already know P(Excellent mark), let's find P(good mark):
There are C_{25}^3 ways to choose the unordered triplet which will represent any possible combination of questions. Now, let's find the number of good triplets of questions:
We have to take two question of of the set of questions which the student knows, there are C_{20}^2 ways to do that, and we have to take one question which the student doesn't know, there are 5 ways to choose that question. So the number of good outcomes is $5 \cdot C_{20}^2$, and P(good mark) is $\dfrac{5 \cdot C_{20}^2}{C_{25}^3}$
So, by subtracting, we get

$$\mathsf{P}(\text{Unsatisfactory mark}) =$$

$$= 1 - \ \mathsf{P}(\text{Excellent mark}) - \ \mathsf{P}(\text{Good mark}) = \boxed{1 - \ \frac{C_{20}^3}{C_{25}^3} - \ \frac{5 \cdot C_{20}^2}{C_{25}^3}}$$

(c) $\mathsf{P}(\text{passing the exam}) = \mathsf{P}(\text{Good mark}) + \mathsf{P}(\text{Excellent mark}) = \boxed{\frac{C_{20}^3 + 5 \cdot C_{20}^2}{C_{25}^3}}$

(d) $\mathsf{P}(\text{passing if she knows the first question}) = 1 - \mathsf{P}(\text{Doesn't know second and third one})$
Let's calculate $\mathsf{P}(\text{Doesn't know second and third one})$:
There are C_5^2 possible ways to choose the last two questions that she doesn't know, and C_{24}^2 to choose the last two questions (24, because we have fixed the one question that she knows out of 25, and don't consider it). So $\mathsf{P}(\text{Doesn't know second and third one}) = \frac{C_5^2}{C_{24}^2}$, and $\mathsf{P}(\text{passing if she knows the first question}) = \boxed{1 - \frac{C_5^2}{C_{24}^2}}$

(e) By definition of conditional probability, we have $\mathsf{P}(\text{knows two out of three} \mid \text{passed})$

$$= \frac{\mathsf{P}(\text{knows two out of three questions } \cap \text{ She passed})}{\mathsf{P}(\text{She passed})}$$

$\mathsf{P}(\text{knows two out of three questions } \cap \text{ She passed}) = \mathsf{P}(\text{good mark}) = \frac{5 \cdot C_{20}^2}{C_{25}^3}$.

$$\mathsf{P}(\text{She passed}) = \frac{C_{20}^3 + 5 \cdot C_{20}^2}{C_{25}^3}$$

So the desired probability is the ratio of these two, which is equal to $\boxed{\frac{5 \cdot C_{20}^2}{C_{20}^3 + 5 \cdot C_{20}^2}}$

Problem 5

The same as previous, but the 3 questions might coincide.

Solution:

Let n be the number of questions, p is the probability that the get the question which we know. Let X be the number of good questions that we get.
Because the questions can be repeated, we have a binomial distribution $X \sim \text{Bin}(n, p)$, which has the probability $\mathsf{P}(X = k) = C_n^k\, p^k\, (1-p)^{n-k}$ for natural k from 0 to n. Here $n = 3$, and also, because we know 20 questions out of 25, $p = \dfrac{4}{5}$.

(a) We want to find $\mathsf{P}(X = 3) = \boxed{\left(\dfrac{4}{5}\right)^3}$

(b)
$$\mathsf{P}(\text{Unsatisfactory mark}) = 1 - \mathsf{P}(\text{Excellent mark}) - \mathsf{P}(\text{Good mark}).$$
We already know P(Excellent mark), let's find P(good mark):
$\mathsf{P}(\text{good mark}) = \mathsf{P}(X = 2) = C_3^2 \left(\dfrac{4}{5}\right)^2 \dfrac{1}{5}$.
So by subtracting, we get $\mathsf{P}(\text{Unsatisfactory mark}) = \boxed{1 - \dfrac{4^2 \cdot 7}{5^3}}$

(c) $\mathsf{P}(\text{passing the exam}) = \mathsf{P}(\text{Good mark}) + \mathsf{P}(\text{Excellent mark}) = \boxed{\dfrac{4^2 \cdot 7}{5^3}}$

(d) $\mathsf{P}(\text{passing if she knows the first question}) = 1 - \mathsf{P}(\text{Doesn't know second and third}) =$
$= 1 - \left(\dfrac{1}{5}\right)^2 = \boxed{\dfrac{24}{25}}$

(e) As in task 4, the desired probability is $\dfrac{\mathsf{P}(\text{Good mark})}{\mathsf{P}(\text{She passed})} = \dfrac{3 \cdot 4^2}{5^3} \cdot \dfrac{5^3}{4^2 \cdot 7} = \boxed{\dfrac{3}{7}}$.

Problem 6

From a brood of mice, containing two white specimens, four mice are taken at random (without return). The probability that both white mice were taken is twice as likely as the probability that neither was taken. How many mice are there in the brood?

Solution:

Let the number of mice in brood be n. Then the probability to take 2 white mice with 2 black ones:

$$\mathsf{P}(2 \text{ white and } 2 \text{ black}) = \frac{C_2^2 C_{n-2}^2}{C_n^4}.$$

Probability to take 4 black mice (and 0 white ones):

$$\mathsf{P}(4 \text{ black}) = \frac{C_2^0 C_{n-2}^4}{C_n^4}.$$

From problem statement we know that

$$\mathsf{P}(2 \text{ white and } 2 \text{ black}) = 2\ \mathsf{P}(4 \text{ black}),$$

which gives an equation on n:

$$\begin{aligned}
\frac{C_2^2 C_{n-2}^2}{C_n^4} &= 2 \cdot \frac{C_2^0 C_{n-2}^4}{C_n^4}, \\
\frac{(n-2)!}{2! \cdot (n-4)!} &= 2 \cdot \frac{(n-2)!}{4! \cdot (n-6)!}, \\
2 \cdot (n-4)! &= 4 \cdot 3 \cdot (n-6)!, \\
(n-4)(n-5) &= 3 \cdot 2, \\
n^2 - 9n + 14 &= 0,
\end{aligned}$$

$$\left[\begin{array}{l} n = 7, \\ n = 2. \end{array}\right.$$

We can't choose 4 mice out of 2, thus, the only fitting answer is $\boxed{n = 7}$.

Problem 7

A player picks a spot at random within a region S on a flat surface. S is split into four sections, each covering 50%, 30%, 12%, and 8% of the total S area. If the chosen spot falls into one of these sections, the player wins a prize with probabilities of 0.01, 0.05, 0.20, and 0.50, respectively. The player has now selected a spot and won a prize. Which section of the S area is the most likely location for the chosen spot?

Solution:

Let's introduce the following events:

$$H_i = \{ \text{ the point hit the } i^{\text{th}} \text{ part of the region } \}, \quad i = 1, 2, 3, 4;$$
$$A = \{ \text{ player won a prize } \}.$$

In order to answer the question we have to calculate the numbers $\mathsf{P}(H_1 \mid A), i = 1, 2, 3, 4$ and to take the maximum. We have

$$\mathsf{P}(H_1) = 0.5, \quad \mathsf{P}(H_2) = 0.3, \quad \mathsf{P}(H_3) = 0.12, \quad \mathsf{P}(H_4) = 0.08,$$

$$\mathsf{P}(A \mid H_1) = 0.01, \quad \mathsf{P}(A \mid H_2) = 0.05, \quad \mathsf{P}(A \mid H_3) = 0.20, \quad \mathsf{P}(A \mid H_4) = 0.50$$

Then

$$\mathsf{P}(A) = \sum_{i=1}^{4} \mathsf{P}(A \mid H_i)\, \mathsf{P}(H_i) = 0.084$$

According to the Bayes' rule, $\mathsf{P}(H_i \mid A) = \dfrac{\mathsf{P}(A \mid H_i)\, \mathsf{P}(H_i)}{\mathsf{P}(A)}, i = 1, 2, 3, 4$. By direct calculations we get

$$\mathsf{P}(H_1 \mid A) = 0.060, \quad \mathsf{P}(H_2 \mid A) = 0.179, \quad \mathsf{P}(H_3 \mid A) = 0.286, \quad \boxed{\mathsf{P}(H_4 \mid A) = 0.475}.$$

So, the fourth part is most likely hitting part of the region.

Problem 8

Lord Wile loves to drink whiskey, the amount of alcohol consumed per day is random, but it is known that he can drink n glasses per day with probability $\frac{1}{n!}e^{-1}$, $n = 0, 1, \ldots$ Yesterday his wife Lady Wile, his son Liddell and his butler decided to kill the lord. If he didn't drink whiskey that day, Lady Wile must have killed him; if he drank exactly one glass, Liddell had the task to commit the murder; otherwise the butler had to do it. Lady Wile is twice as likely to resort to poisoning as to strangulation; the butler, in contrast, chooses strangulation with twice the likelihood of poisoning; and Liddell is equally likely to choose any of these methods. Despite all efforts, there is no guarantee that Lord Wile will surely die as a result of any of the attempts to kill him, however, he is three times more likely to become a victim of strangulation than poisoning.

Lord Wile is dead today. What is the probability that the butler killed him?

Solution:

Let's denote events, used in problem statement:

W – lady Wile attempted to murder lord Wile,

L – Liddell attempted to murder lord Wile,

B – butler attempted to murder lord Wile,

S – attempt to murder via strangulation,

P – attempt to murder via poisoning,

M – lord Wile was indeed murdered.

From number of glasses with consumed alcohol:

$$\mathsf{P}(W) \overset{n=0}{=} \frac{1}{e}, \qquad \mathsf{P}(L) \overset{n=1}{=} \frac{1}{e}, \qquad \mathsf{P}(B) \overset{n\geq 1}{=} 1 - \frac{2}{e}.$$

We know the following information about methods of murder in relation to actors:

$$\mathsf{P}(P \mid W) = 2\ \mathsf{P}(S \mid W), \qquad \mathsf{P}(S \mid B) = 2\ \mathsf{P}(P \mid B), \qquad \mathsf{P}(P \mid L) = \mathsf{P}(S \mid L),$$

and since P and S are the only methods of murder, they constitute full sample space. Thus, from total probability:

$$\mathsf{P}(P \mid W) = \frac{2}{3}, \qquad \mathsf{P}(P \mid B) = \frac{1}{3}, \qquad \mathsf{P}(P \mid L) = \frac{1}{2},$$
$$\mathsf{P}(S \mid W) = \frac{1}{3}, \qquad \mathsf{P}(S \mid B) = \frac{2}{3}, \qquad \mathsf{P}(S \mid L) = \frac{1}{2}.$$

Another condition is that murder is thrice more probable via strangulation than via poisoning:

$$\mathsf{P}(M \mid S) = 3\ \mathsf{P}(M \mid P),$$

but since those probabilities have different sample space, in contrast to previous block, we can't assign numerical values to them. Let's denote $\mathsf{P}(M \mid P)$ as p. Then:

$$\mathsf{P}(M \mid P) = p, \qquad \mathsf{P}(M \mid S) = 3p.$$

We want to find $\mathsf{P}(B \mid M)$. Applying Bayes theorem:

$$\mathsf{P}(B \mid M) = \frac{\mathsf{P}(M \mid B) \cdot \mathsf{P}(B)}{\mathsf{P}(M)} = \frac{\mathsf{P}(M \mid B) \cdot \mathsf{P}(B)}{\mathsf{P}(M \mid B) \cdot \mathsf{P}(B) + \mathsf{P}(M \mid L) \cdot \mathsf{P}(L) + \mathsf{P}(M \mid W) \cdot \mathsf{P}(W)}.$$

Using total probability decomposition for conditional probabilities:

$$\begin{aligned} \mathsf{P}(M \mid B) &= \mathsf{P}(M \mid S) \cdot \mathsf{P}(S \mid B) + \mathsf{P}(M \mid P) \cdot \mathsf{P}(P \mid B), \\ \mathsf{P}(M \mid L) &= \mathsf{P}(M \mid S) \cdot \mathsf{P}(S \mid L) + \mathsf{P}(M \mid P) \cdot \mathsf{P}(P \mid L), \\ \mathsf{P}(M \mid W) &= \mathsf{P}(M \mid S) \cdot \mathsf{P}(S \mid W) + \mathsf{P}(M \mid P) \cdot \mathsf{P}(P \mid W). \end{aligned}$$

Applying everything into $\mathsf{P}(B \mid M)$ we get:

$$\begin{aligned} \mathsf{P}(B \mid M) &= \frac{\left(3p \cdot \frac{2}{3} + p \cdot \frac{1}{3}\right) \cdot \left(1 - \frac{2}{e}\right)}{\left(3p \cdot \frac{2}{3} + p \cdot \frac{1}{3}\right) \cdot \left(1 - \frac{2}{e}\right) + \left(3p \cdot \frac{1}{2} + p \cdot \frac{1}{2}\right) \cdot \frac{1}{e} + \left(3p \cdot \frac{1}{3} + p \cdot \frac{2}{3}\right) \cdot \frac{1}{e}} = \\ &= \boxed{\frac{7(e-2)}{7e-3} \approx 31.4\%}. \end{aligned}$$