Quiz

(a) Three shooters fired their guns, with two bullets hitting the target. Find the probability that the third shooter has hit the target if the probabilities of hitting the target by the first, second and third shooters are 0.6, 0.5, and 0.4, respectively.

(b) If a coin with probability of tail equal to $2/3$ is tossed 5 times, find the probability of at least 4 heads.

Solution:

(a) Let S_1, S_2 and S_3 be events that respective shooter has hit the target. Event $2/3$ means that 2 of 3 shooters have hit the mark. Using definition of conditional probability:

$$\mathsf{P}(S_3 \mid 2/3) = \frac{\mathsf{P}(S_3 \cap 2/3)}{\mathsf{P}(2/3)}.$$

Probability that 2 out of 3 shooter hit the target and one of them was 3[rd] one:

$$\begin{aligned}\mathsf{P}(S_3 \cap 2/3) &= \mathsf{P}(S_3 \cap S_1 \cap \overline{S}_2) + \mathsf{P}(S_3 \cap S_2 \cap \overline{S}_1) = \\ &= \mathsf{P}(S_3)\mathsf{P}(S_1)(1 - \mathsf{P}(S_2)) + \mathsf{P}(S_3)\mathsf{P}(S_2)(1 - \mathsf{P}(S_1)) = \\ &= 0.4 \cdot 0.6 \cdot 0.5 + 0.4 \cdot 0.5 \cdot 0.4 = 0.2.\end{aligned}$$

Probability that any 2 out of 3 shooter hit the target:

$$\begin{aligned}\mathsf{P}(2/3) &= \mathsf{P}(S_3 \cap 2/3) + \mathsf{P}(\overline{S}_3 \cap 2/3) = \mathsf{P}(S_3 \cap 2/3) + \mathsf{P}(S_2 \cap S_1 \cap \overline{S}_3) = \\ &= \mathsf{P}(S_3 \cap 2/3) + \mathsf{P}(S_2)\mathsf{P}(S_1)(1 - \mathsf{P}(S_3)) = \\ &= 0.2 + 0.5 \cdot 0.6 \cdot 0.6 = 0.38.\end{aligned}$$

Thus, the required probability:

$$\mathsf{P}(S_3 \mid 2/3) = \frac{0.2}{0.38} = \boxed{\frac{10}{19}}.$$

(b) Let X be a number heads after 5 Bernoulli trials. Since probability of tail is $\frac{2}{3}$, the probability of head is $p = \frac{1}{3}$. Thus, $X \sim \text{Bin}\left(5, \frac{1}{3}\right)$.

P.m.f. of binomial variable $X \sim \text{Bin}\,(n, p)$ is $\mathsf{P}(X = k) = C_n^k p^k (1 - p)^{n-k}$.

We want to find $\mathsf{P}(X \geq 4) = \mathsf{P}(X = 4) + \mathsf{P}(X = 5)$, which gives:

$$\mathsf{P}(X = 4) + \mathsf{P}(X = 5) = C_5^4 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^5 = \boxed{\frac{11}{243}}.$$

Problem 1

A number is chosen at random from set $S = \{-1, 0, 1\}$. Let X be the number chosen. Find the expected value, variance, and standard deviation of X.

Solution:

Each value in the set S is equiprobable:

X	-1	0	1
P_X	$1/3$	$1/3$	$1/3$

It means that the probability distribution is symmetrical with respect to the point 0, and the expected value is $\mathsf{E}(X) = \boxed{0}$.
The variance is calculated by definition:

$$\mathsf{V}(X) = \sum_{x \in X} (x - \mathsf{E}(X))^2 \, \mathsf{P}_X(x) = (-1-0)^2 \cdot \frac{1}{3} + (0-0)^2 \cdot \frac{1}{3} + (1-0)^2 \cdot \frac{1}{3} = \boxed{\frac{2}{3}}.$$

Standard deviation of X by definition is the square root of the variance:

$$\sigma(X) = \sqrt{\mathsf{V}(X)} = \boxed{\sqrt{\frac{2}{3}}}.$$

Problem 2

A random variable X has the following distribution:

X	0	1	2	4
P_X	1/3	1/3	1/6	1/6

Find $\mathsf{E}(X)$, $\mathsf{E}\left(X(X+1)\right)$, $\mathsf{V}(X)$ and $\sigma(X)$.

Solution:

Let's calculate expected value by definition:

$$\mathsf{E}(X) = \sum_{x \in X} x \, \mathsf{P}_X(x) = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} = \boxed{\frac{4}{3}}.$$

Due to linearity of expected value:

$$\mathsf{E}\left(X(X+1)\right) = \mathsf{E}\left(X^2 + X\right) = \mathsf{E}\left(X^2\right) + \mathsf{E}(X).$$

Here we need to find $\mathsf{E}\left(X^2\right)$. By definition of the expected value of the function:

$$\mathsf{E}\left(X^2\right) = \sum_{x \in X} x^2 \, \mathsf{P}_X(x) = 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} = \frac{11}{3}.$$

Substituting into equation for $\mathsf{E}\left(X(X+1)\right)$:

$$\mathsf{E}\left(X(X+1)\right) = \frac{11}{3} + \frac{4}{3} = \boxed{5}.$$

The variance via $\mathsf{E}\left(X^2\right)$ and $\mathsf{E}(X)$:

$$\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 = \frac{11}{3} - \left(\frac{4}{3}\right)^2 = \boxed{\frac{17}{9}}.$$

Standard deviation is the square root of the variance:

$$\sigma(X) = \sqrt{\mathsf{V}(X)} = \boxed{\frac{\sqrt{17}}{3}}.$$

Problem 3

X is a random variable with $\mathsf{E}(X) = 100$ and $\mathsf{V}(X) = 15$. Find

(a) $\mathsf{E}(X^2)$.

(b) $\mathsf{E}(3X + 10)$.

(c) $\mathsf{E}(-X)$.

(d) $\mathsf{V}(-X)$.

(e) $\sigma(-X)$.

Solution:

(a) Since $\mathsf{V}(X) = \mathsf{E}(X^2) - \mathsf{E}(X)^2$:

$$\mathsf{E}\left(X^2\right) = \mathsf{V}(X) + \mathsf{E}(X)^2 = 15 + 100^2 = \boxed{10015}.$$

(b) Due to linearity of expected value:

$$\mathsf{E}(3X + 10) = 3\,\mathsf{E}(X) + 10 = 3 \cdot 100 + 10 = \boxed{310}.$$

(c) Due to linearity of expected value:

$$\mathsf{E}(-X) = -\mathsf{E}(X) = \boxed{-100}.$$

(d) Since $\mathsf{V}(aX + b) = a^2\,\mathsf{V}(X)$:

$$\mathsf{V}(-X) = (-1)^2\,\mathsf{V}(X) = \mathsf{V}(X) = \boxed{15}.$$

(e) Standard deviation is the square root of the variance:

$$\sigma(-X) = \sqrt{\mathsf{V}(-X)} = \boxed{\sqrt{15}}.$$

Problem 4

A coin is tossed three times. Let X be the number of heads that turn up. Find $\mathsf{V}(X)$ and $\sigma(X)$.

Solution:

The variable X is distributed binomially, since we have repeating experiment. The number of trials is $n = 3$, the probability of success (head turns up) is $p = 1/2$:

$$X \sim \text{Bin}\left(3, \frac{1}{2}\right).$$

Probability of k successes in binomial distribution is given by the formula $\mathsf{P}_X(X = k) = C_n^k p^k (1-p)^{n-k}$. Using this knowledge we can construct probability mass function for X:

X	P_X
0	$C_3^0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{8}$
1	$C_3^1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^2 = \frac{3}{8}$
2	$C_3^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^1 = \frac{3}{8}$
3	$C_3^3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^0 = \frac{1}{8}$

By definition of expected value, and using the distribution of X:

$$\mathsf{E}(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3}{2}.$$

$$\mathsf{E}\left(X^2\right) = 0^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{3}{8} + 2^2 \cdot \frac{3}{8} + 3^2 \cdot \frac{1}{8} = 3.$$

The variance then:

$$\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 = 3 - \left(\frac{3}{2}\right)^2 = \boxed{\frac{3}{4}}.$$

Standard deviation is the square root of the variance:

$$\sigma(X) = \sqrt{\mathsf{V}(X)} = \boxed{\frac{\sqrt{3}}{2}}.$$

Problem 5

A box contains 10 white balls and 2 black balls. 6 balls are selected at random. Random variable X is equal to number of black balls in the 6 selected.

(a) Find the distribution of random variable X.

(b) Find expected value $\mathsf{E}(X)$.

(c) Find expected value of X, given that you were told that $X > 0$ ($\mathsf{E}(X \mid X > 0)$).

Solution:

(a) Probability to take 0 black balls and 6 white balls is:

$$\mathsf{P}(X = 0) = \frac{C_2^0\, C_{10}^6}{C_{12}^6} = \frac{5}{22}.$$

The rest of probabilities is calculated similarly:

$$\mathsf{P}(X = 1) = \frac{C_2^1\, C_{10}^5}{C_{12}^6} = \frac{6}{11}, \qquad \mathsf{P}(X = 2) = \frac{C_2^2\, C_{10}^4}{C_{12}^6} = \frac{5}{22}.$$

Thus, the probability distribution is

X	0	1	2
P_X	$5/22$	$6/11$	$5/22$

(b) Since the probability distribution is symmetrical with respect to $X = 1$, expected value is $\mathsf{E}(X) = \boxed{1}$.

(c) Using total probability (events "$X > 0$" and "$X = 0$" are mutually exclusive):

$$\mathsf{E}(X) = \mathsf{E}(X \mid X > 0) \cdot \mathsf{P}(X > 0) + \mathsf{E}(X \mid X = 0) \cdot \mathsf{P}(X = 0).$$

The expected value $\mathsf{E}(X \mid X = 0)$ obviously equals 0. Then:

$$\mathsf{E}(X \mid X > 0) = \frac{\mathsf{E}(X)}{\mathsf{P}(X > 0)}.$$

Probability $\mathsf{P}(X > 0)$ consists of $\mathsf{P}(X = 1)$ and $\mathsf{P}(X = 2)$:

$$\mathsf{P}(X > 0) = \mathsf{P}(X = 1) + \mathsf{P}(X = 2) = \frac{6}{11} + \frac{5}{22} = \frac{17}{22}.$$

Then the required expectation:

$$\mathsf{E}(X \mid X > 0) = \frac{1}{17/22} = \boxed{\frac{22}{17}}.$$