Seminar 5

Quiz

(a) Consider the following p.m.f. of a random variable X:

x	1789	1790	1791	1792	1793
$\mathsf{P}_X(x)$	1/5	1/4	1/10	1/4	1/5

Find the following **WITHOUT CALCULATOR**:

i) $\mathsf{E}(X)$,

ii) $\mathsf{V}(X)$.

(b) X is a random variable with $\mathsf{E}\left(X^2\right) = 3.6$ and $\mathsf{P}(X = 2) = 0.6$ and $\mathsf{P}(X = 3) = 0.1$. X takes just one other value between 0 and 3. Find the variance of X.

Solution:

(a) i) Using the symmetry of p.m.f., $\mathsf{E}(X)$ is a central value of p.m.f.: $\boxed{\mathsf{E}(X) = 1791}$.

ii) Two possible good solutions, identical in essence:

1) By definition of $\mathsf{V}(X)$:

$$\mathsf{V}(X) = \mathsf{E}\left(X - \mathsf{E}(X)\right)^2 = \sum_{x \in X} (x - 1791)^2 \cdot \mathsf{P}_X(x) =$$
$$= (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{10} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{5} =$$
$$= \frac{4}{5} \cdot 2 + \frac{1}{4} \cdot 2 = \boxed{\frac{21}{10}}.$$

2) Using property $\mathsf{V}(X + c) = \mathsf{V}(X)$, where c is a constant.
Let's construct new random variable $X^* = X - 1791$. We know from the given property that $\mathsf{V}\left(X^*\right) = \mathsf{V}(X)$, but the p.m.f. of X^* is easier to deal with:

x	-2	-1	0	1	2
$\mathsf{P}_{X^*}(x)$	1/5	1/4	1/10	1/4	1/5

Also from linearity of expected value and symmetry of p.m.f. $\mathsf{E}(X^*) = 0$, which gives $\mathsf{V}(X^*) = \mathsf{E}\left(X^{*2}\right)$:

$$\mathsf{V}(X) = \mathsf{V}\left(X^*\right) = \sum_{x \in X^*} x^2 \cdot \mathsf{P}_{X^*}(x) =$$
$$= (-2)^2 \cdot \frac{1}{5} + (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{10} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{1}{5} =$$
$$= \frac{4}{5} \cdot 2 + \frac{1}{4} \cdot 2 = \boxed{\frac{21}{10}}.$$

(b) Let unknown value of X between 0 and 3 be k. Since the total probability for all values of X is 1, probability of $X = k$ is $\mathsf{P}_X(k) = 1 - 0.6 - 0.1 = 0.3$.

By definition of $\mathsf{E}(X^2)$:

$$\mathsf{E}\left(X^2\right) = \sum_{x \in X} x^2 \cdot \mathsf{P}_X(x) = 2^2 \cdot 0.6 + 3^2 \cdot 0.1 + k^2 \cdot 0.3 = 3.6.$$

$$k^2 = \frac{3.6 - 2.4 - 0.9}{0.3} = 12 - 8 - 3 = 1.$$

Since $k \in [0, 3)$:

$$k = 1.$$

Expected value $\mathsf{E}(X)$ by definition:

$$\mathsf{E}(X) = \sum_{x \in X} x \cdot \mathsf{P}_X(x) = 2 \cdot 0.6 + 3 \cdot 0.1 + 1 \cdot 0.3 = 1.8.$$

Finally, variance $\mathsf{V}(X)$:

$$\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 = 3.6 - 1.8^2 = \boxed{0.36}.$$

Problem 1

A random variable X has a binomial distribution with mean 10 and variance 6.
Find $\mathsf{P}(X = 4)$.

Solution:

Firstly, let's find $\mathsf{E}(X)$ and $\mathsf{V}(X)$ for $X \sim \text{Bin}(n, p)$.
X is a sum of n independent Bernoulli trials $Y_i \sim \text{Bernoulli}(p)$ with probability of success p:

$$X = \sum_{i=1}^{n} Y_i.$$

Expected value of Y_i by definition:

$$\mathsf{E}(Y_i) = \sum_{y \in Y_i} y \cdot \mathsf{P}_{Y_i}(y) = 1 \cdot p + 0 \cdot (1 - p) = p.$$

Then using linearity of expected value, $\mathsf{E}(X)$ is following:

$$\mathsf{E}(X) = \mathsf{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathsf{E}(Y_i) = \sum_{i=1}^{n} p = np.$$

Using same logic, expected value of Y_i^2 is:

$$\mathsf{E}\left(Y_i^2\right) = \sum_{y \in Y_i} y^2 \cdot \mathsf{P}_{Y_i}(y) = 1^2 \cdot p + 0^2 \cdot (1 - p) = p.$$

And the variance of Y_i:

$$\mathsf{V}(Y_i) = \mathsf{E}\left(Y_i^2\right) - \mathsf{E}(Y_i)^2 = p - p^2 = p(1 - p).$$

Since Y_i are collectively independent:

$$\mathsf{V}(X) = \mathsf{V}\left(\sum_{i=1}^{n} Y_i\right) \stackrel{\text{ind}}{=} \sum_{i=1}^{n} \mathsf{V}(Y_i) = \sum_{i=1}^{n} p(1 - p) = np(1 - p).$$

Now, when we know $\mathsf{E}(X)$ and $\mathsf{V}(X)$, we can find n and p:

$$\begin{cases} np = 10, \\ np(1 - p) = 6, \end{cases} \implies \begin{cases} 1 - p = \dfrac{3}{5}, \\ np = 10, \end{cases} \implies \begin{cases} p = \dfrac{2}{5}, \\ n = 25. \end{cases}$$

The p.m.f. for $X \sim \mathrm{Bin}(n, p)$ is following:

$$\mathsf{P}(X = k) = C_n^k p^k (1-p)^{n-k},$$

which for $k = 4, n = 25, p = \dfrac{2}{5}$ gives:

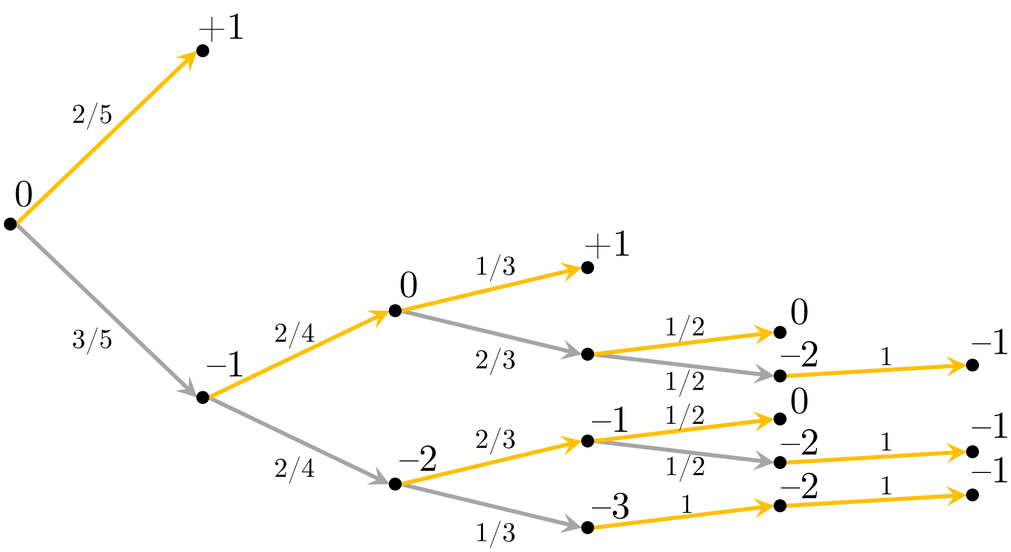
$$\mathsf{P}(X = 4) = C_{25}^4 \left(\frac{2}{5}\right)^4 \left(\frac{3}{5}\right)^{21} = \boxed{\frac{2^5 \cdot 3^{21} \cdot 11 \cdot 23}{5^{23}} \approx 0.71\%}.$$

Problem 2

A box contains two gold balls and three silver balls. You are allowed to choose successively balls from box at random. You win 1 dollar each time you draw a gold ball and lose 1 dollar each time you draw a silver ball. After a draw, the ball is not replaced. Show that, if you draw until you are ahead by 1 dollar or until there are no more gold balls, this is a favorable game.

Solution:

Let's construct probability tree, using strategy from problem statement. Drawing gold and silver balls are illustrated with respective arrows. Each draw is accompanied by its probability. Nodes of the tree represent current fortune of the player.



Let X be the player's fortune at the end of the game. From probability tree:

$$\mathsf{P}(X=+1)=\frac{2}{5}+\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}=\frac{1}{2},$$

$$\mathsf{P}(X=0)=\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}+\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}=\frac{1}{5},$$

$$\mathsf{P}(X=-1)=\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot 1+\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot 1+\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}\cdot 1\cdot 1=\frac{3}{10}.$$

By the definition of expected value:

$$\mathsf{E}(X)=1\cdot\frac{1}{2}+0\cdot\frac{1}{5}+(-1)\cdot\frac{3}{10}=\boxed{\frac{1}{5}>0}.$$

Thus, the player will earn on average $1/5$ of a dollar.

Problem 3

Consider two random variables with the following joint distribution:

$X \setminus Y$	1	2
3	1/4	1/4
5	1/6	1/3

(a) Find the marginal distributions of X and of Y.

(b) Are X and Y independent?

(c) Find $\mathsf{E}(X + 2Y)$, $\mathsf{E}(XY)$, $\mathsf{V}(X + Y)$.

(d) Suppose that random variables U and V have same distributions as X and Y, but are independent. Find the joint distribution of U and V.

Solution:

(a) Marginal distributions of X and Y from the joint table:

X	3	5
P_X	1/2	1/2

Y	1	2
P_Y	5/12	7/12

(b) X and Y are independent if their joint distribution equals to the product of marginal ones, $\forall (x, y) \in (X, Y)$:

$$\mathsf{P}_{X,Y}(x, y) = \mathsf{P}_X(x) \cdot \mathsf{P}_Y(y).$$

For example, let $x = 3$ and $y = 1$:

$$\left.\begin{aligned} \mathsf{P}_{X,Y}(3, 1) &= \frac{1}{4}, \\ \mathsf{P}_X(3) \cdot \mathsf{P}_Y(1) &= \frac{1}{2} \cdot \frac{5}{12} = \frac{5}{24}, \end{aligned}\right. \quad \Longrightarrow \quad \mathsf{P}_{X,Y}(3, 1) \neq \mathsf{P}_X(3) \cdot \mathsf{P}_Y(1).$$

Thus, X and Y are $\boxed{\text{not independent}}$.

(c) From linearity of expected value:

$$\mathsf{E}(X + 2Y) = \mathsf{E}(X) + 2\ \mathsf{E}(Y).$$

By definition of expected value:

$$\mathsf{E}(X) = \sum_{x \in X} x \cdot \mathsf{P}_X(x) = 3 \cdot \frac{1}{2} + 5 \cdot \frac{1}{2} = 4,$$
$$\mathsf{E}(Y) = \sum_{y \in Y} y \cdot \mathsf{P}_Y(y) = 1 \cdot \frac{5}{12} + 2 \cdot \frac{7}{12} = \frac{19}{12},$$

which gives:

$$\mathsf{E}(X + 2Y) = 4 + 2 \cdot \frac{19}{12} = \boxed{\frac{43}{6}}.$$

$\mathsf{E}(XY)$ can not be represented as product of expected values $\mathsf{E}(X) \cdot \mathsf{E}(Y)$, since X and Y are dependent. That's why we have to calculate $\mathsf{E}(XY)$ by definition:

$$\mathsf{E}(XY) = \sum_{(x,y) \in (X,Y)} xy \cdot \mathsf{P}_{X,Y}(x,y) = 3 \cdot 1 \cdot \frac{1}{4} + 3 \cdot 2 \cdot \frac{1}{4} + 5 \cdot 1 \cdot \frac{1}{6} + 5 \cdot 2 \cdot \frac{1}{3} = \boxed{\frac{77}{12}}.$$

$\mathsf{V}(X+Y)$ can not be represented as $\mathsf{V}(X)+\mathsf{V}(Y)$ for similar reasons. It can be calculated by definition or using identity:

$$\mathsf{V}(X+Y) = \mathsf{V}(X) + \mathsf{V}(Y) + 2\ \mathrm{Cov}(X,Y).$$

Let's use identity. Variances require expectations of squares:

$$\mathsf{E}\left(X^2\right) = \sum_{x \in X} x^2 \cdot \mathsf{P}_X(x) = 3^2 \cdot \frac{1}{2} + 5^2 \cdot \frac{1}{2} = 17,$$
$$\mathsf{E}\left(Y^2\right) = \sum_{y \in Y} y^2 \cdot \mathsf{P}_Y(y) = 1^2 \cdot \frac{5}{12} + 2^2 \cdot \frac{7}{12} = \frac{11}{4}.$$

Variances then:

$$\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 = 17 - 4^2 = 1,$$
$$\mathsf{V}(Y) = \mathsf{E}\left(Y^2\right) - \mathsf{E}(Y)^2 = \frac{11}{4} - \left(\frac{19}{12}\right)^2 = \frac{35}{144}.$$

Covariance is found as follows:

$$\mathrm{Cov}(X,Y) = \mathsf{E}(XY) - \mathsf{E}(X) \cdot \mathsf{E}(Y) = \frac{77}{12} - 4 \cdot \frac{19}{12} = \frac{1}{12}.$$

Values for covariance and variances give:

$$\mathsf{V}(X+Y) = 1 + \frac{35}{144} + 2 \cdot \frac{1}{12} = \boxed{\frac{203}{144}}.$$

(d) Since marginal distributions of U and V coincide with those of X and Y respectively, we can construct the joint distribution table of U and V by element-wise multiplication of marginal probabilities:

$U \setminus V$	1	2
3	5/24	7/24
5	5/24	7/24

Problem 4

Suppose that X and Y have the following joint probability mass function:

$X \setminus Y$	1	2	3
1	0.25	0.25	0
2	0	0.25	0.25

What is the correlation coefficient?

Solution:

Correlation coefficient $\text{Corr}(X, Y)$ is defined as follows:

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma(X) \cdot \sigma(Y)}.$$

Covariance $\text{Cov}(X, Y)$ can be found via following identity:

$$\text{Cov}(X, Y) = \mathsf{E}(XY) - \mathsf{E}(X) \cdot \mathsf{E}(Y).$$

In order to find $\mathsf{E}(X)$ and $\mathsf{E}(Y)$, we should construct marginal distributions of X and Y:

X	1	2
P_X	0.5	0.5

Y	1	2	3
P_Y	0.25	0.5	0.25

Having zeros in joint distributions, it's clear that X and Y are not independent. By definition of expected value:

$$\mathsf{E}(X) = \sum_{x \in X} x \cdot \mathsf{P}_X(x) = 1 \cdot 0.5 + 2 \cdot 0.5 = 1.5,$$

$$\mathsf{E}(Y) = \sum_{y \in Y} y \cdot \mathsf{P}_Y(y) = 1 \cdot 0.25 + 2 \cdot 0.5 + 3 \cdot 0.25 = 2,$$

$$\mathsf{E}(XY) = \sum_{(x,y) \in (X,Y)} xy \cdot \mathsf{P}_{X,Y}(x, y) = 1 \cdot 1 \cdot 0.25 + 1 \cdot 2 \cdot 0.25 + \\ + 1 \cdot 3 \cdot 0 + 2 \cdot 1 \cdot 0 + 2 \cdot 2 \cdot 0.25 + 2 \cdot 3 \cdot 0.25 = 3.25.$$

Covariance then:

$$\text{Cov}(X, Y) = 3.25 - 1.5 \cdot 2 = \frac{1}{4}.$$

In order to find standard deviations $\sigma(X)$ and $\sigma(Y)$, we should calculate respective variances, which in turn require expectations of squares:

$$\mathsf{E}\left(X^2\right) = \sum_{x \in X} x^2 \cdot \mathsf{P}_X(x) = 1^2 \cdot 0.5 + 2^2 \cdot 0.5 = 2.5,$$
$$\mathsf{E}\left(Y^2\right) = \sum_{y \in Y} y^2 \cdot \mathsf{P}_Y(y) = 1^2 \cdot 0.25 + 2^2 \cdot 0.5 + 3^2 \cdot 0.25 = 4.5.$$

Variances then:

$$\mathsf{V}(X) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 = 2.5 - 1.5^2 = 0.25,$$
$$\mathsf{V}(Y) = \mathsf{E}\left(Y^2\right) - \mathsf{E}(Y)^2 = 4.5 - 2^2 = 0.5,$$

and standard deviations:

$$\sigma(X) = \sqrt{\mathsf{V}(X)} = \sqrt{0.25} = \frac{1}{2},$$
$$\sigma(Y) = \sqrt{\mathsf{V}(Y)} = \sqrt{0.5} = \frac{1}{\sqrt{2}}.$$

Thus, the correlation coefficient is following:

$$\text{Corr}(X, Y) = \frac{\frac{1}{4}}{\frac{1}{2} \cdot \frac{1}{\sqrt{2}}} = \boxed{\frac{1}{\sqrt{2}}}.$$

Problem 5

The probability distribution of a random variable X is:

$X = x$	-1	0	1
$\mathsf{P}(X = x)$	a	b	a

What is the correlation coefficient between X and X^2?

Solution:

Firstly, let's calculate covariance between X and X^2. We will require expectations $\mathsf{E}(X)$, $\mathsf{E}(X^2)$ and $\mathsf{E}(X^3)$:

$$\begin{aligned}
\mathsf{E}(X) &= \sum_{x \in X} x \cdot \mathsf{P}_X(x) = -1 \cdot a + 0 \cdot b + 1 \cdot a = 0, \\
\mathsf{E}\left(X^2\right) &= \sum_{x \in X} x^2 \cdot \mathsf{P}_X(x) = (-1)^2 \cdot a + 0^2 \cdot b + 1^2 \cdot a = 2a, \\
\mathsf{E}\left(X^3\right) &= \sum_{x \in X} x^3 \cdot \mathsf{P}_X(x) = (-1)^3 \cdot a + 0^3 \cdot b + 1^3 \cdot a = 0.
\end{aligned}$$

Covariance then:

$$\operatorname{Cov}\left(X, X^2\right) = \mathsf{E}\left(X^3\right) - \mathsf{E}(X) \cdot \mathsf{E}\left(X^2\right) = 0 - 0 \cdot 2a = 0.$$

Since covariance is 0, correlation coefficient is also 0:

$$\operatorname{Corr}\left(X, X^2\right) = \boxed{0}.$$

This example brings up a very important difference between independence and uncorrelatedness:

$$\begin{gathered}
\text{If variables are independent} \Rightarrow \text{they are uncorrelated.} \\
\text{If variables are uncorrelated} \not\Rightarrow \text{they are independent.}
\end{gathered}$$

In our problem variables are clearly dependent, but still uncorrelated. This can happen since correlation is a measure of **linear** dependency between variables, and square function is non-linear. Thus, the following is true for any constants a and b:

$$\operatorname{Corr}(X, aX + b) = \operatorname{sgn} a.$$