Quiz

Joint distribution of 2 random variables is set in the table below:

$X \setminus Y$	0	1	2
0	0.2	0.2	0.4
1	0.05	a	b

(a) Find a and b such that random variables X and Y are independent.

(b) Find $\mathsf{E}(XY)$ with those values of a and b.

Solution:

(a) X and Y are independent if their joint distribution equals to the product of marginal ones, $\forall (x, y) \in (X, Y)$:

$$\mathsf{P}_{X,Y}(x, y) = \mathsf{P}_X(x) \cdot \mathsf{P}_Y(y).$$

Marginal distributions for X and Y:

X	0	1
P_X	0.8	0.2

Y	0	1	2
P_Y	0.25	$0.2 + a$	$0.4 + b$

We know that $\mathsf{P}_X(1) = 0.2$ since total probability of marginal distribution of X is 1.

For $x = 1$ and $y = 1$:

$$\begin{aligned} \mathsf{P}_X(1) \cdot \mathsf{P}_Y(1) &= 0.2 \cdot (0.2 + a), \\ \mathsf{P}_{X,Y}(1, 1) &= a, \end{aligned} \quad \Longrightarrow \quad 0.2 \cdot (0.2 + a) = a,$$

which yields the only result for a and $b = 0.2 - a - 0.05$ (from total probability):

$$\boxed{\begin{aligned} a &= 0.05, \\ b &= 0.1. \end{aligned}}$$

We should check that all combinations of x and y satisfy the independence condition, which is true.

(b) The fastest way to calculate $\mathsf{E}(XY)$ is by definition, since 4 out of 6 products of x and y are zeroes:

$$\mathsf{E}(XY) = \sum_{(x,y) \in (X,Y)} xy \cdot \mathsf{P}_{X,Y}(x, y) = 1 \cdot 1 \cdot 0.05 + 1 \cdot 2 \cdot 0.1 = \boxed{0.25}.$$

Problem 1

There are two independent fair coin tosses. Let random variables X and Y be the following:

(a)

X − number of heads,
Y − indicator of an event that both heads and tails were in experiment.

(b)

X − number of heads,
Y − number of tails.

Construct conditional p.m.f.-s $\mathsf{P}_{X|Y}(x \mid y)$ for those cases.

Solution:

(a) Let's find out what values of X and Y correspond to elementary outcomes.

	HH	HT	TH	TT
$X = \#H$	2	1	1	0
$Y = I_{\{H,T\}}$	0	1	1	0

Joint distribution $\mathsf{P}_{X,Y}(x, y)$ then:

$X \setminus Y$	0	1
0	$\frac{1}{4}$	0
1	0	$\frac{1}{2}$
2	$\frac{1}{4}$	0

Marginal distributions:

X	0	1	2
P_X	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Y	0	1
P_Y	$\frac{1}{2}$	$\frac{1}{2}$

Using the definition of conditional distribution:

$$\mathsf{P}_{X|Y}(x \mid y) = \frac{\mathsf{P}_{X,Y}(x,y)}{\mathsf{P}_Y(y)}$$

X	0	1	2
$\mathsf{P}_{X\|Y=0}$	$\frac{1}{2}$	0	$\frac{1}{2}$

X	0	1	2
$\mathsf{P}_{X\|Y=1}$	0	1	0

(b) Again with values of X and Y:

	HH	HT	TH	TT
$X = \#H$	2	1	1	0
$Y = \#T$	0	1	1	2

Joint distribution $\mathsf{P}_{X,Y}(x,y)$ then:

$X \setminus Y$	0	1	2
0	0	0	$\frac{1}{4}$
1	0	$\frac{1}{2}$	0
2	$\frac{1}{4}$	0	0

Marginal distributions:

X	0	1	2
P_X	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Y	0	1	2
P_Y	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Conditional distributions:

X	0	1	2
$\mathsf{P}_{X\|Y=0}$	0	0	1

X	0	1	2
$\mathsf{P}_{X\|Y=1}$	0	1	0

X	0	1	2
$\mathsf{P}_{X\|Y=2}$	1	0	0

Problem 2

Consider two random variables X and Y. They both take the values 0, 1 and 2. The joint probabilities for each pair are given by the following table.

$Y \setminus X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

(a) Calculate marginal distributions, expected values and covariance of X and Y.

(b) Calculate covariance of the random variables X and V, where $V = X - Y$.

(c) Calculate $\mathsf{E}(X \mid Y = 0)$ and $\mathsf{E}(X \mid V = 1)$.

(d) The random variable W has the same marginal distribution as X and the random variable Z has the same distribution as Y. It is also known that W and Z are independent. Write down the table for the joint probabilities of W and Z.

Solution:

(a) Marginal distributions of X and Y from the joint table:

X	0	1	2
P_X	0.4	0.3	0.3

Y	0	1	2
P_Y	0.4	0.3	0.3

Expected values from marginal distributions:

$$\mathsf{E}(X) = \mathsf{E}(Y) = 1 \cdot 0.3 + 2 \cdot 0.3 = \boxed{0.9}.$$

The covariance of X and Y is calculated as a difference between $\mathsf{E}(XY)$ and $\mathsf{E}(X)\,\mathsf{E}(Y)$. Let's calculate $\mathsf{E}(XY)$ using the joint distribution table (only 4 out 9 probabilities are assigned to non-zero values of X and Y, and 2 out 4 leftover probabilities are not zeros themselves):

$$\mathsf{E}(XY) = 2 \cdot 1 \cdot 0.1 + 1 \cdot 2 \cdot 0.1 = 0.4.$$

Covariance then is:

$$\mathrm{Cov}(X, Y) = \mathsf{E}(XY) - \mathsf{E}(X)\,\mathsf{E}(Y) = 0.4 - 0.9 \cdot 0.9 = \boxed{-0.41}.$$

(b) Using the linearity of expected value:

$$\begin{aligned}\operatorname{Cov}(X,V) &= \operatorname{Cov}(X, X-Y) = \mathsf{E}(X(X-Y)) - \mathsf{E}(X)\,\mathsf{E}(X-Y) = \\ &= \mathsf{E}\left(X^2\right) - \mathsf{E}(XY) - \mathsf{E}(X)\left(\mathsf{E}(X) - \mathsf{E}(Y)\right) = \\ &= \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 - \left(\mathsf{E}(XY) - \mathsf{E}(X)\,\mathsf{E}(Y)\right) = \mathsf{V}(X) - \operatorname{Cov}(X,Y).\end{aligned}$$

So, basically, the only thing we need to calculate is $\mathsf{E}\left(X^2\right)$:

$$\mathsf{E}\left(X^2\right) = 1^2 \cdot 0.3 + 2^2 \cdot 0.3 = 1.5.$$

Thus, the covariance of X and V:

$$\operatorname{Cov}(X,V) = \mathsf{E}\left(X^2\right) - \mathsf{E}(X)^2 - \operatorname{Cov}(X,Y) = 1.5 - 0.9^2 - (-0.41) = \boxed{1.1}.$$

(c) By the definition of conditional expected value:

$$\mathsf{E}(X \mid Y=0) = \sum_x x\,\mathsf{P}(X=x \mid Y=0) = \sum_x x\,\frac{\mathsf{P}(X=x, Y=0)}{\mathsf{P}(Y=0)}.$$

$Y \setminus X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

From marginal distribution of Y $\mathsf{P}(Y=0) = 0.4$ and all possible values of X with respective probabilities are taken from the green-lighted cells of the joint distribution table.

$$\mathsf{E}(X \mid Y=0) = \frac{1}{\mathsf{P}(Y=0)} \sum_x x\,\mathsf{P}(X=x, Y=0) = \frac{1}{0.4}\left(1 \cdot 0.2 + 2 \cdot 0.2\right) = \boxed{1.5}.$$

Identically for $\mathsf{E}(X \mid V=1)$:

$$\mathsf{E}(X \mid V=1) = \frac{1}{\mathsf{P}(V=1)} \sum_x x\,\mathsf{P}(X=x, V=1).$$

Let's find out, which probabilities from the original joint table represent $V = X - Y = 1$. Possible points are: $(X=1, Y=0)$ and $(X=2, Y=1)$.

The total probability for $\mathsf{P}(V=1) = 0.2 + 0.1 = 0.3$, and the expected value will take 2 colored probabilities in its calculation.

$$\mathsf{E}(X \mid V=1) = \frac{1}{\mathsf{P}(V=1)} \sum_x x\,\mathsf{P}(X=x, V=1) = \frac{1}{0.3}\left(1 \cdot 0.2 + 2 \cdot 0.1\right) = \boxed{\frac{4}{3}}.$$

$Y \setminus X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

(d) Variables are considered to be independent if their joint probability distribution is equal to the product of their marginal distributions:

$$\mathsf{P}(W = w, Z = z) = \mathsf{P}(W = w) \cdot \mathsf{P}(Z = z).$$

Since marginal distributions of W and Z coincide with those of X and Y respectively, we can construct the joint distribution table of W and Z by element-wise multiplication of marginal probabilities:

$Z \setminus W$	0	1	2
0	0.16	0.12	0.12
1	0.12	0.09	0.09
2	0.12	0.09	0.09

Problem 3

The p.d.f. of Y is $g(y) = d \cdot y^{-4}$, $1 < y < \infty$.

(a) Find d.

(b) Find c.d.f.

(c) Find $\mathsf{E}(Y)$.

(d) Find m such that $\mathsf{P}(Y > m) = 0.5$.

(e) Find $\mathsf{P}\left(Y > \mathsf{E}(Y)\right)$.

Solution:

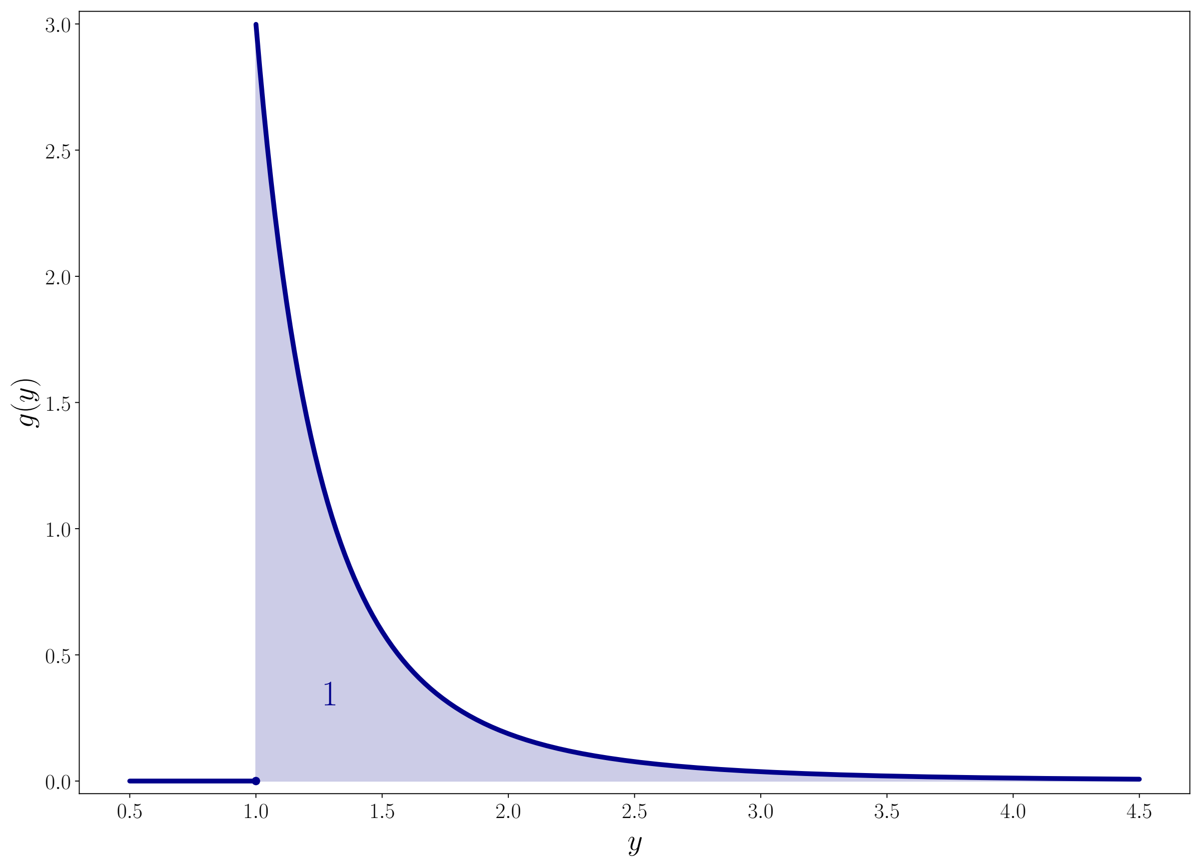

Figure 1: Probability density function $g(y)$.

(a) Using normalization condition (see fig. 1):

$$\int\limits_{-\infty}^{\infty} g(y)dy = 1, \qquad \Longrightarrow \qquad \int\limits_{-\infty}^{1} 0 \cdot dy + \int\limits_{1}^{\infty} d \cdot y^{-4} dy = 1, \qquad \Longrightarrow \qquad d = \frac{1}{\int\limits_{1}^{\infty} y^{-4} dy}.$$

$$d = \frac{1}{-\left.\dfrac{y^{-3}}{3}\right|_{1}^{\infty}} = \frac{1}{0 - \left(-\dfrac{1}{3}\right)} = \boxed{3}.$$

(b) The c.d.f. $G(y)$ is found from p.d.f. $g(y)$ following way:

$$G(y) = \mathsf{P}(Y \le y) = \int\limits_{-\infty}^{y} g(\eta) d\eta = \left[\begin{array}{ll} \int\limits_{-\infty}^{1} 0 \cdot d\eta + \int\limits_{1}^{y} 3\eta^{-4} d\eta = -\eta^{-3}\Big|_{1}^{y}, & y > 1, \\ \int\limits_{-\infty}^{y} 0 \cdot d\eta = 0, & y \le 1, \end{array}\right. =$$

$$= \boxed{\left(1 - y^{-3}\right) \cdot I_{\{y>1\}}}.$$

The resulting expression for $G(y)$, shown in fig. 2, satisfies main properties of c.d.f.:

$$\lim_{y \to -\infty} G(y) = 0, \qquad \lim_{y \to +\infty} G(y) = 1.$$

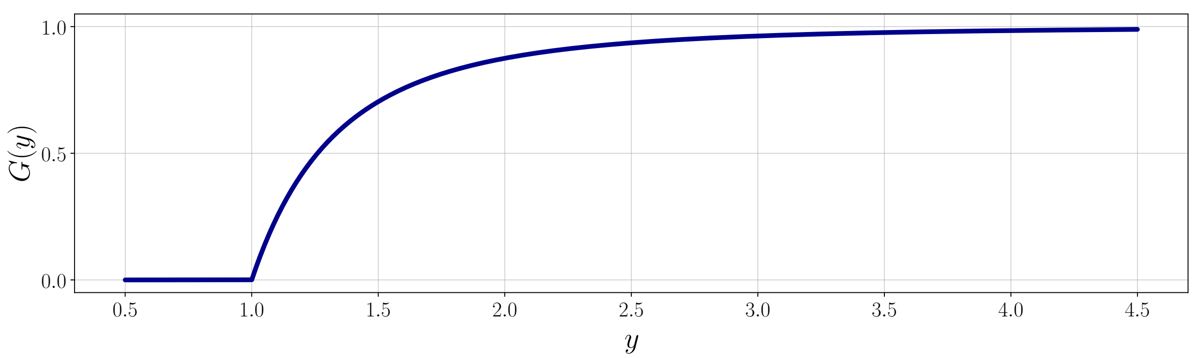

Figure 2: Cumulative distribution function $G(y)$.

(c) By definition of expected value:

$$\mathsf{E}(Y) = \int\limits_{-\infty}^{\infty} y g(y) dy = \int\limits_{-\infty}^{1} y \cdot 0 \cdot dy + \int\limits_{1}^{\infty} y \cdot 3y^{-4} dy = -\frac{3}{2} y^{-2}\Big|_{1}^{\infty} = 0 - \left(-\frac{3}{2}\right) = \boxed{\frac{3}{2}}.$$

(d) From total probability it's obvious that:

$$\mathsf{P}(Y > m) = \mathsf{P}(Y \le m) = 0.5.$$

Such value m is called median of random variable Y and separates lower half of probability distribution from higher one. By definition of c.d.f. $G(y)$:

$$G(m) = \mathsf{P}(Y \le m) = 0.5.$$

$$1 - m^{-3} = 0.5 \qquad \Longrightarrow \qquad m = \boxed{\sqrt[3]{2}}.$$

Geometric definition of median m is shown in fig. 3. Since $g(y)$ is positively-skewed, mean $\mathsf{E}(Y)$ is greater than median m.

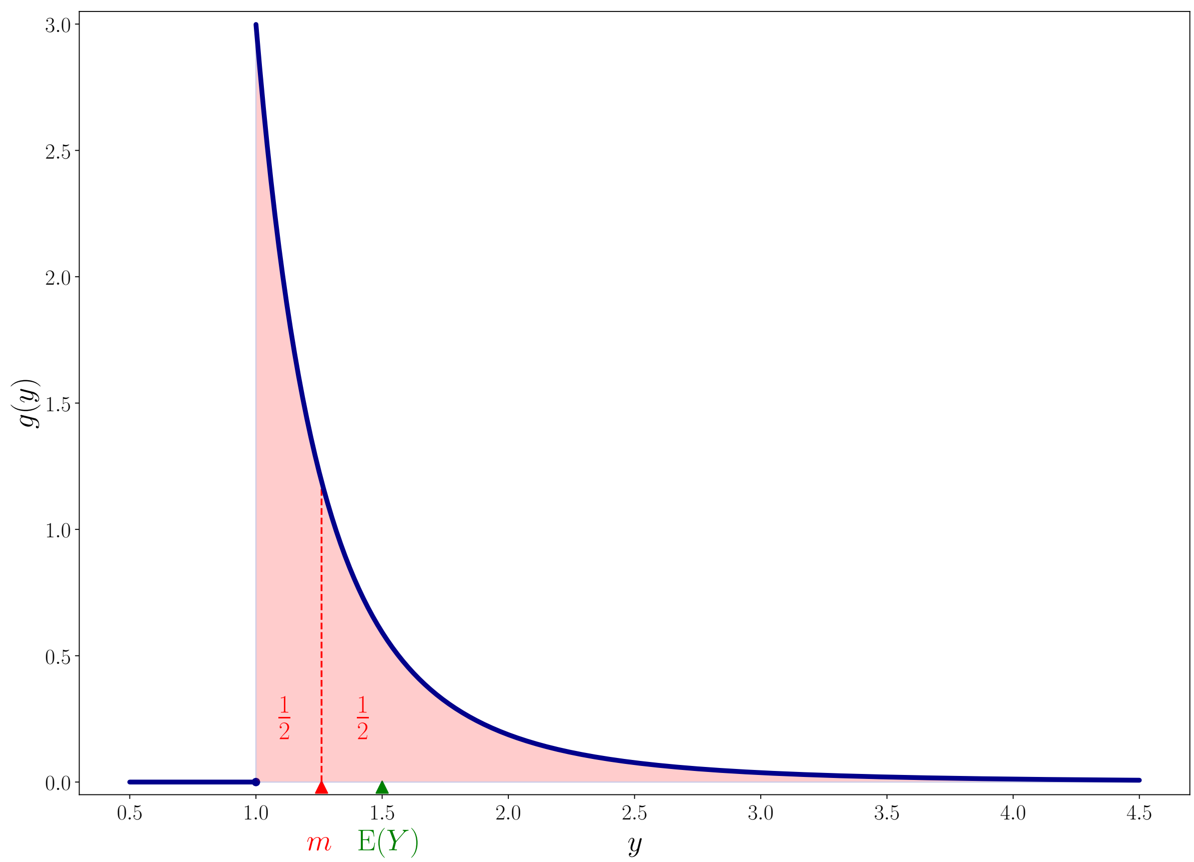

Figure 3: Median m in p.d.f. $g(y)$.

(e) Using definition of c.d.f.:

$$\mathsf{P}\left(Y > \mathsf{E}(Y)\right) = 1 - \mathsf{P}\left(Y \le \mathsf{E}(Y)\right) = 1 - G(\mathsf{E}(Y)) = 1 - G\left(\frac{3}{2}\right) =$$

$$= 1 - \left(1 - \left(\frac{3}{2}\right)^{-3}\right) = \boxed{\left(\frac{2}{3}\right)^{3}}.$$

Geometric meaning of $\mathsf{P}\left(Y > \mathsf{E}(Y)\right)$ is shown in fig. 4.

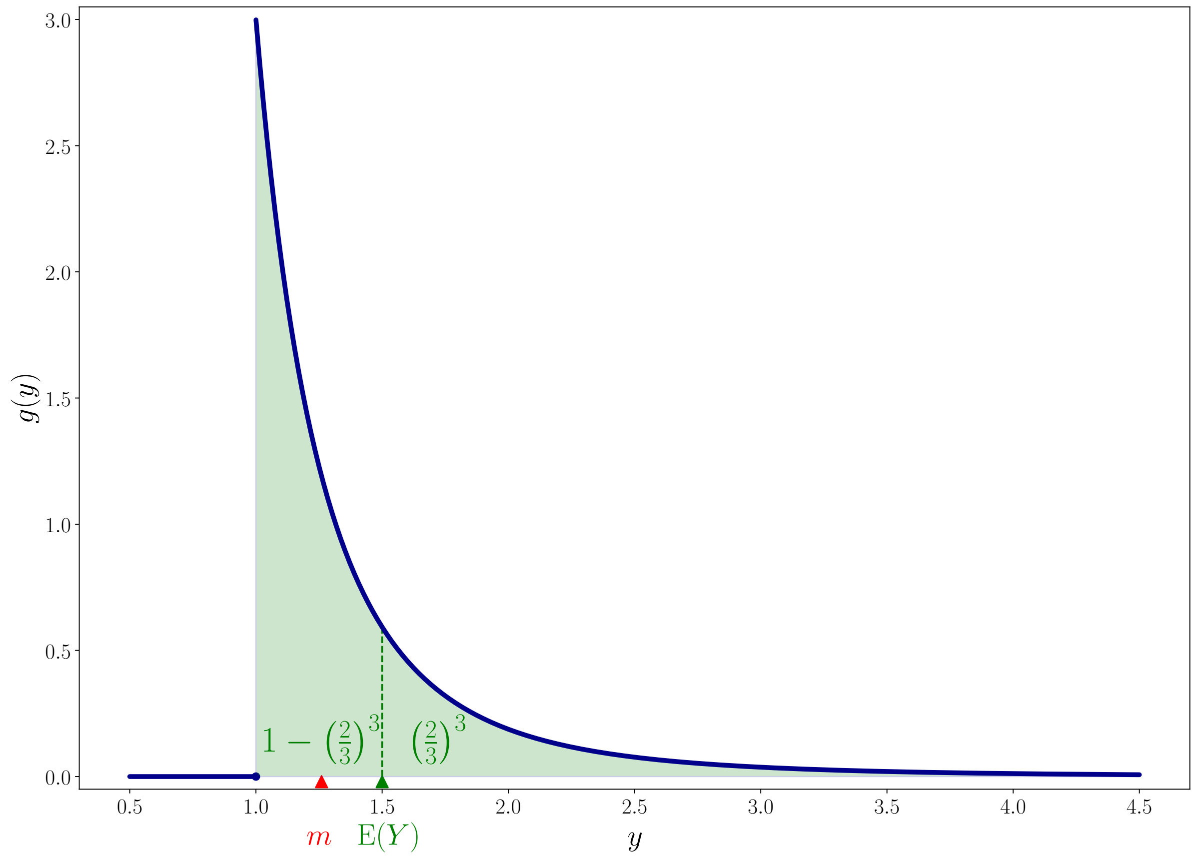

Figure 4: $\mathsf{P}\left(Y > \mathsf{E}(Y)\right)$ in p.d.f. $g(y)$.

Areas from figures 3 and 4 can be shown directly as points of c.d.f. $G(y)$, which is illustrated in fig. 5.

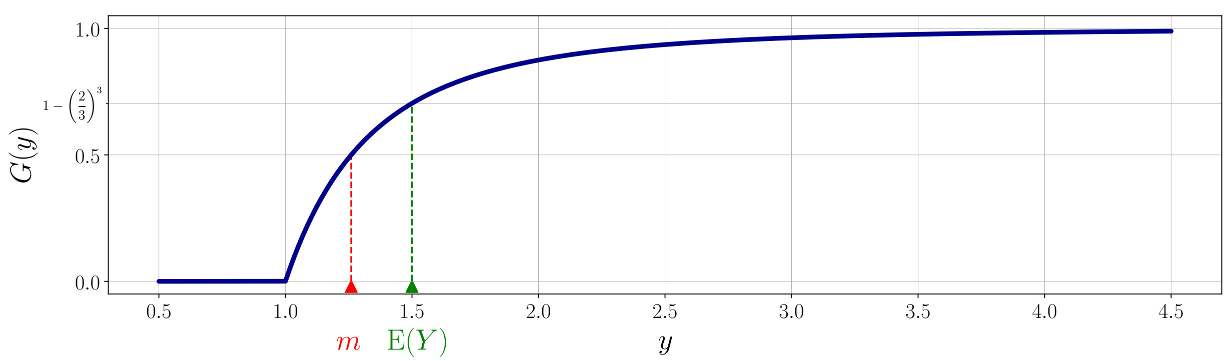

Figure 5: Median m and mean $\mathsf{E}(Y)$ in c.d.f. $G(y)$.

Problem 4

Let X be a random variable with uniform distribution on the interval $(-2, 3)$.
Find $\mathsf{P}(X > 2 \mid X > 1)$.

Solution:

By definition of conditional probability:

$$\mathsf{P}(X > 2 \mid X > 1) = \frac{\mathsf{P}(X > 2 \cap X > 1)}{\mathsf{P}(X > 1)} = \frac{\mathsf{P}(X > 2)}{\mathsf{P}(X > 1)}.$$

There are 2 possible solutions, both require p.d.f. $f(x)$ of X.

For $X \sim \mathcal{U}(a, b)$ p.d.f. is $f(x) = \dfrac{1}{b-a} \cdot I_{\{a \leq x \leq b\}}$. Thus, for $X \sim \mathcal{U}(-2, 3)$ p.d.f. is:

$$f(x) = \frac{1}{5} \cdot I_{\{-2 \leq x \leq 3\}}.$$

(a) Geometric solution.

Measure of $X > 1$ is twice as big as measure of $X > 2$, which is clear from fig. 6.

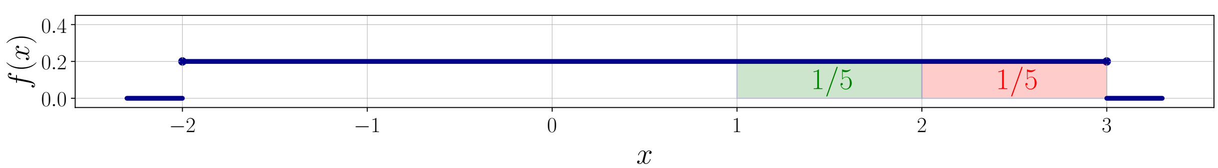

Figure 6: Probability density function $f(x)$.

Thus, required probability is:

$$\mathsf{P}(X > 2 \mid X > 1) = \frac{\mathsf{P}(X > 2)}{\mathsf{P}(X > 1)} = \frac{1/5}{1/5 + 1/5} = \boxed{\frac{1}{2}}.$$

(b) Solution with c.d.f.

If we didn't know that areas under p.d.f. $f(x)$ are easy to calculate, we would find probabilities $\mathsf{P}(X > 1)$ and $\mathsf{P}(X > 2)$ via c.d.f. $F(x)$:

$$F(x) = \mathsf{P}(X \leq x) = \int\limits_{-\infty}^{x} f(\xi) d\xi =$$

$$= \begin{bmatrix} \int\limits_{-\infty}^{-2} 0 \cdot d\xi + \int\limits_{-2}^{3} \frac{1}{5} \cdot d\xi + \int\limits_{3}^{x} 0 \cdot d\xi, \quad x > 3, \\ \int\limits_{-\infty}^{-2} 0 \cdot d\xi + \int\limits_{-2}^{x} \frac{1}{5} \cdot d\xi, \quad -2 \leq x \leq 3, \\ \int\limits_{-\infty}^{x} 0 \cdot d\xi, \quad x < -2, \end{bmatrix} =$$

$$= \begin{bmatrix} 0 + \left.\frac{\xi}{5}\right|_{-2}^{3} + 0, \quad x > 3, \\ 0 + \left.\frac{\xi}{5}\right|_{-2}^{x}, \quad -2 \leq x \leq 3, \\ 0, \quad x < -2, \end{bmatrix} = \begin{bmatrix} 1, \quad x > 3, \\ \frac{x+2}{5}, \quad -2 \leq x \leq 3, \\ 0, \quad x < -2. \end{bmatrix}$$

C.d.f. $F(x)$ is illustrated in fig. 7.

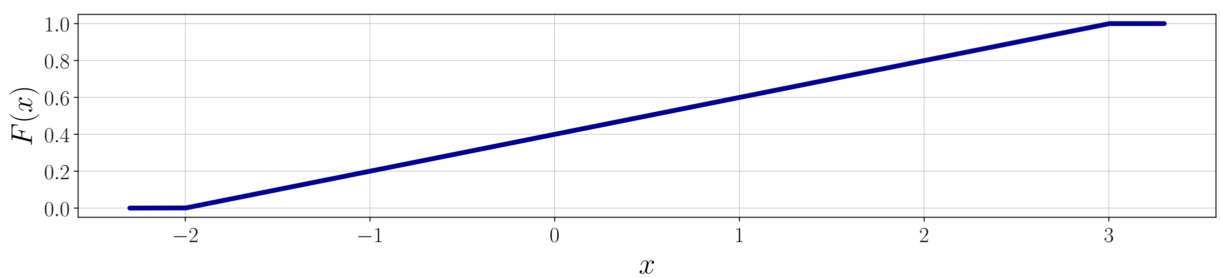

Figure 7: Cumulative distribution function $F(x)$.

Since $\mathsf{P}(X > x) = 1 - \mathsf{P}(X \leq x) = 1 - F(x)$, required probability is:

$$\mathsf{P}(X > 2 \mid X > 1) = \frac{\mathsf{P}(X > 2)}{\mathsf{P}(X > 1)} = \frac{1 - F(2)}{1 - F(1)} = \frac{1 - 0.8}{1 - 0.6} = \frac{0.2}{0.4} = \boxed{\frac{1}{2}}.$$