Problem 1

Suppose $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$, $Y \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ and $\rho(X, Y) = 0.8$. Find the joint distribution of X and Y.

Solution:

Definition of correlation coefficient:

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{\mathsf{E}(XY) - \mathsf{E}(X) \cdot \mathsf{E}(Y)}{\sigma_X \sigma_Y}$$

Moments of Bernoulli variables:

$$X \sim \text{Bernoulli}(p) \quad \Longrightarrow \quad \mathsf{E}(X) = p, \quad \mathsf{V}(X) = p(1-p),$$

thus

$$\mathsf{E}(X) = \mathsf{E}(Y) = \frac{1}{2},$$

$$\mathsf{V}(X) = \mathsf{V}(Y) = \frac{1}{4} \quad \Longrightarrow \quad \sigma_X \sigma_Y = \frac{1}{4}.$$

Finding $\mathsf{E}(XY)$ from $\rho(X, Y)$:

$$\mathsf{E}(XY) = \rho(X, Y) \cdot \sigma_X \sigma_Y + \mathsf{E}(X) \cdot \mathsf{E}(Y) = \frac{8}{10} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} = \frac{9}{20}.$$

By "definition" of $\mathsf{E}(XY)$:

$$\mathsf{E}(XY) = \sum_{x,y \in (X,Y)} x \cdot y \cdot \mathsf{P}_{X,Y}(x, y) = 0 + 0 + 0 + 1 \cdot 1 \cdot \mathsf{P}_{X,Y}(1, 1),$$

$$\Longrightarrow \quad \mathsf{P}_{X,Y}(1, 1) = \frac{9}{20}.$$

Adding probabilities to get $\frac{1}{2}$ in marginals, we get following p.m.f.:

$X \setminus Y$	0	1
0	9/20	1/20
1	1/20	9/20

Problem 2

Suppose random variables X and Y have joint normal distribution.

(a) If X and Y are standard normal, and $\mathsf{P}(X+Y > 1.96) = 0.025$, what is the correlation between X and Y ?

(b) If X and Y are independent, what is $\mathsf{P}(X > 1.96 \mid |Y| > 1.96)$?

Solution:

(a) The sum of normal random variables is also normal:

$$X \sim \mathcal{N}(0,1), \quad Y \sim \mathcal{N}(0,1) \quad \Longrightarrow \quad X+Y \sim \mathcal{N}\left(0, \sigma^2\right),$$

$$\begin{aligned}\sigma^2 = \mathsf{V}(X) + \mathsf{V}(Y) + 2\,\mathrm{Cov}(X,Y) &= \mathsf{V}(X) + \mathsf{V}(Y) + 2\rho\sqrt{\mathsf{V}(X)\cdot\mathsf{V}(Y)} = \\ &= 1 + 1 + 2\rho\cdot 1\cdot 1 = 2 + 2\rho.\end{aligned}$$

Using standardization:

$$\mathsf{P}(X+Y > 1.96) = \mathsf{P}\left(Z > \frac{1.96}{\sqrt{2+2\rho}}\right) = 1 - \Phi\left(\frac{1.96}{\sqrt{2+2\rho}}\right) = 0.025,$$

$$\Phi\left(\frac{1.96}{\sqrt{2+2\rho}}\right) = 0.975,$$

$$\frac{1.96}{\sqrt{2+2\rho}} \approx 1.96,$$

$$\boxed{\rho \approx -\frac{1}{2}}.$$

(b) Since X and Y are independent:

$$\mathsf{P}(X > 1.96 \mid |Y| > 1.96) = \mathsf{P}(X > 1.96) = \boxed{0.025}.$$

Problem 3

Let two random variables have joint p.d.f.:

$$f(x,y) = \begin{cases} cxy, & 0 < x, y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find c, marginal p.d.f. f_X, marginal p.d.f. f_Y. Are X and Y independent?

(b) Find $\mathrm{Cov}(X, Y)$.

(c) Let $g(x) = \mathsf{E}(Y \mid X = x)$. Find $g(x)$.

(d) Find $\mathsf{P}(X^2 > Y^2)$.

Solution:

(a) Using normalization condition:

$$\iint\limits_{-\infty}^{\infty} f(x,y)dxdy = \int\limits_0^2\int\limits_0^2 cxydxdy = c\int\limits_0^2 xdx\int\limits_0^2 ydy = 1,$$

$$c = \frac{1}{\int\limits_0^2 xdx \int\limits_0^2 ydy} = \frac{1}{\left.\frac{x^2}{2}\right|_0^2 \cdot \left.\frac{y^2}{2}\right|_0^2} = \frac{1}{\frac{2^2}{2}\cdot\frac{2^2}{2}} = \boxed{\frac{1}{4}}.$$

P.d.f. $f(x,y)$ is illustrated in fig. 1 and 2.

Marginal distributions by definition:

$$f_X(x) = \int\limits_{-\infty}^{\infty} f(x,y)dy = \int\limits_0^2 \frac{1}{4}xydy \cdot I_{\{0<x<2\}} = \frac{1}{4}x \cdot I_{\{0<x<2\}} \cdot \left.\frac{y^2}{2}\right|_0^2 = \boxed{\frac{x}{2}\cdot I_{\{0<x<2\}}}.$$

$$f_Y(y) = \int\limits_{-\infty}^{\infty} f(x,y)dx = \int\limits_0^2 \frac{1}{4}xydx \cdot I_{\{0<y<2\}} = \frac{1}{4}y \cdot I_{\{0<y<2\}} \cdot \left.\frac{x^2}{2}\right|_0^2 = \boxed{\frac{y}{2}\cdot I_{\{0<y<2\}}}.$$

X and Y are considered to be independent if $\forall(x,y) \in (X,Y) : f_X(x)\cdot f_Y(y) = f(x,y)$:

$$f_X(x)\cdot f_Y(y) = \frac{x}{2}\cdot I_{\{0<x<2\}}\cdot\frac{y}{2}\cdot I_{\{0<y<2\}} = \frac{1}{4}xy\cdot I_{\{0<x,y<2\}} = f(x,y).$$

Thus, X and Y are $\boxed{\text{independent}}$.

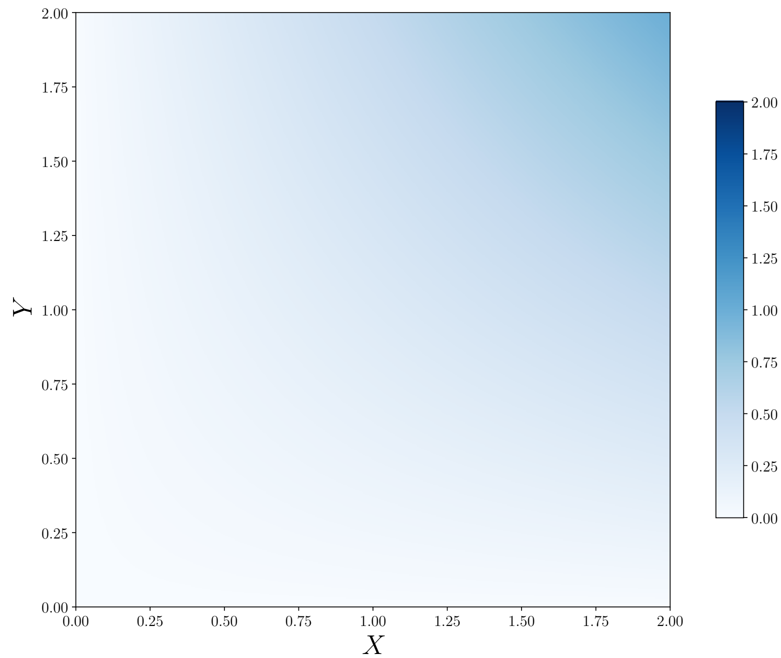

Figure 1: Probability density function $f(x,y) = \frac{1}{4}xy \cdot I_{\{0<x,y<2\}}$ (top view).

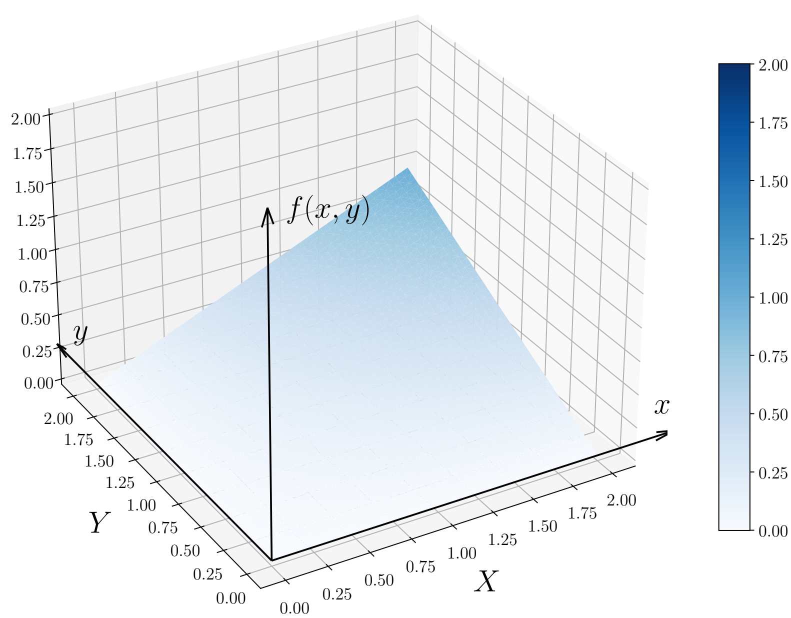

Figure 2: Probability density function $f(x,y) = \frac{1}{4}xy \cdot I_{\{0<x,y<2\}}$ (side view).

(b) Since X and Y are independent, their covariance is zero:

$$\mathrm{Cov}(X, Y) = \boxed{0}.$$

(c) By definition of conditional expectation:

$$\mathsf{E}(Y \mid X = x) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y \mid x) dy.$$

But since Y is independent from X, its conditional p.d.f. is equal to marginal one:

$$\forall (x, y) \in (X, Y): \quad f_{Y|X}(y \mid x) = f_Y(y).$$

Substitution of f_Y into conditional expectation $\mathsf{E}(Y \mid X = x)$ gives marginal $\mathsf{E}(Y)$:

$$g(x) = \mathsf{E}(Y \mid X = x) = \mathsf{E}(Y) = \int_{-\infty}^{\infty} y \cdot f_Y(y) dy = \int_0^2 y \cdot \frac{y}{2} dy = \left.\frac{y^3}{6}\right|_0^2 = \boxed{\frac{4}{3}}.$$

(d) Since X and Y both take only positive values:

$$\mathsf{P}\left(X^2 > Y^2\right) \equiv \mathsf{P}\left(X > Y\right).$$

From the geometry of $f(x, y)$ in fig. 1, it's clear that line $Y = X$ separates two identical volumes under p.d.f., which means that $\mathsf{P}\left(X > Y\right) = \mathsf{P}\left(X < Y\right) = \frac{1}{2}$. We can prove it explicitly via integration:

$$\mathsf{P}\left(X > Y\right) = \int_0^2 \left(\int_0^x \frac{1}{4} xy dy \right) dx = \int_0^2 \frac{1}{4} x \cdot \left.\frac{y^2}{2}\right|_0^x \cdot dx = \int_0^2 \frac{x^3}{8} dx = \left.\frac{x^4}{32}\right|_0^2 = \boxed{\frac{1}{2}}.$$

Problem 4

Same as in **Problem 3**, but joint p.d.f. is:

$$f(x,y) = \begin{cases} cxy, & 0 < y < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Solution:

(a) Using normalization condition:

$$\iint\limits_{-\infty}^{\infty} f(x,y)dxdy = \int\limits_0^2 \left(\int\limits_0^x cxydy \right) dx = c \int\limits_0^2 x \left(\int\limits_0^x ydy \right) dx = 1,$$

$$c = \frac{1}{\int\limits_0^2 x \left(\int\limits_0^x ydy \right) dx} = \frac{1}{\int\limits_0^2 x \cdot \left. \frac{y^2}{2} \right|_0^x \cdot dx} = \frac{1}{\int\limits_0^2 \frac{x^3}{2} dx} = \frac{1}{\left. \frac{x^4}{8} \right|_0^2} = \boxed{\frac{1}{2}}.$$

P.d.f. $f(x,y)$ is illustrated in fig. 3 and 4.

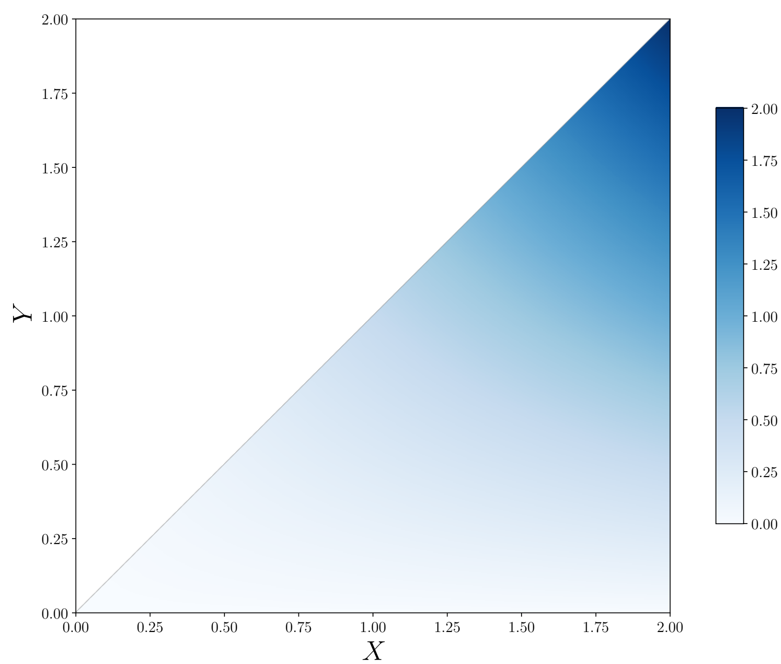

Figure 3: Probability density function $f(x,y) = \frac{1}{2}xy \cdot I_{\{0<y<x<2\}}$ (top view).

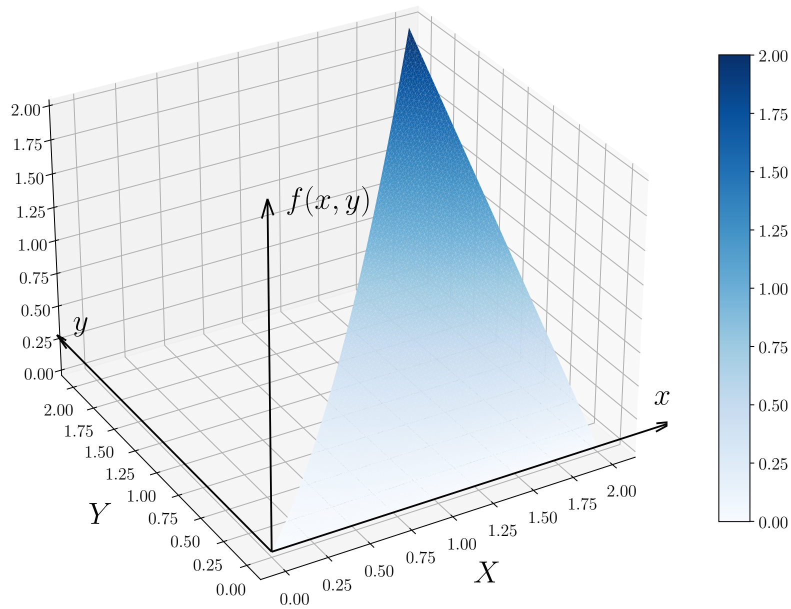

Figure 4: Probability density function $f(x,y) = \dfrac{1}{2}xy \cdot I_{\{0<y<x<2\}}$ (side view).

Marginal distributions by definition:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \int_0^x \frac{1}{2}xydy \cdot I_{\{0<x<2\}} = \frac{1}{2}x \cdot I_{\{0<x<2\}} \cdot \frac{y^2}{2}\Big|_0^x = \boxed{\frac{x^3}{4} \cdot I_{\{0<x<2\}}}.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \int_y^2 \frac{1}{2}xydx \cdot I_{\{0<y<2\}} = \frac{1}{2}y \cdot I_{\{0<y<2\}} \cdot \frac{x^2}{2}\Big|_y^2 = \boxed{\left(y - \frac{y^3}{4}\right) \cdot I_{\{0<y<2\}}}.$$

In this case $f(x,y)$ is clearly not equal to $f_X(x) \cdot f_Y(y)$, which means that X and Y are not independent.

Interestingly, marginal distribution $f_X(x)$ is monotonically increasing, while $f_Y(y)$ has maximum in $y = \dfrac{2}{\sqrt{3}}$, in contrary to marginal distributions from Problem 3, in which both of them were symmetric and monotonic. This is illustrated in fig. 5 and 6.

(b) Since $\mathrm{Cov}(X,Y) = \mathsf{E}(XY) - \mathsf{E}(X) \cdot \mathsf{E}(Y)$, we have to calculate expectations first:

$$\mathsf{E}(X) = \int_{-\infty}^{\infty} xf_X(x)dx = \int_0^2 x \cdot \frac{x^3}{4}dx = \frac{x^5}{20}\Big|_0^2 = \frac{8}{5}.$$

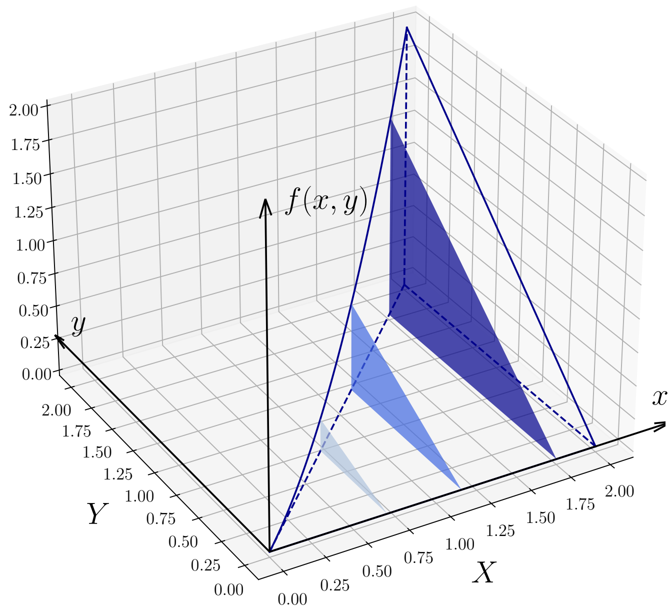

Figure 5: Marginal slices of p.d.f. $f(x, y)$ over X-axis.

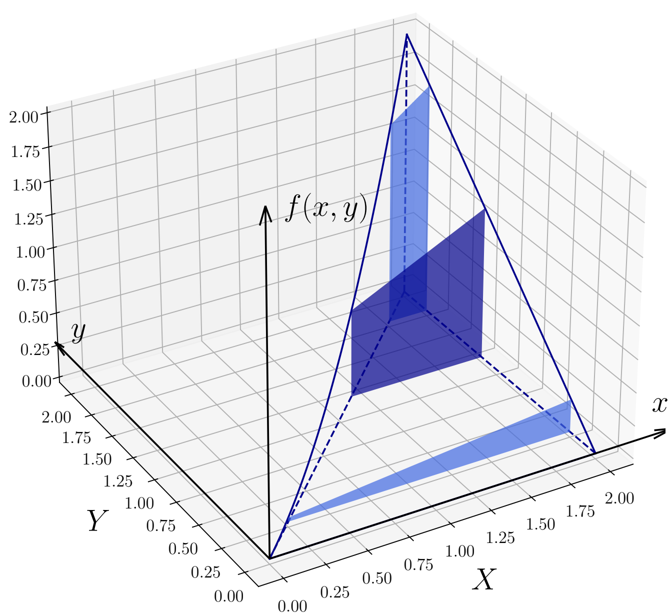

Figure 6: Marginal slices of p.d.f. $f(x, y)$ over Y-axis.

$$\mathsf{E}(Y)=\int\limits_{-\infty}^{\infty} y f_Y(y)dy=\int\limits_0^2 y\cdot\left(y-\frac{y^3}{4}\right)dy=\left.\frac{y^3}{3}\right|_0^2-\left.\frac{y^5}{20}\right|_0^2=\frac{8}{3}-\frac{8}{5}=\frac{16}{15}.$$

$$\mathsf{E}(XY)=\iint\limits_{-\infty}^{\infty} xyf(x,y)dxdy=\int\limits_0^2\left(\int\limits_0^x xy\cdot\frac{1}{2}xydy\right)dx=\frac{1}{2}\int\limits_0^2 x^2\left(\int\limits_0^x y^2dy\right)dx=$$
$$=\frac{1}{2}\int\limits_0^2 x^2\cdot\left.\frac{y^3}{3}\right|_0^x dx=\frac{1}{2}\int\limits_0^2\frac{x^5}{3}dx=\frac{1}{2}\cdot\left.\frac{x^6}{18}\right|_0^2=\frac{16}{9}.$$

Covariance then:

$$\mathrm{Cov}(X,Y)=\frac{16}{9}-\frac{8}{5}\cdot\frac{16}{15}=\boxed{\frac{16}{225}}.$$

(c) We need conditional p.d.f. $f_{Y|X}(y\mid x)$. By definition:

$$f_{Y|X}(y\mid x)=\frac{f(x,y)}{f_X(x)}=\frac{\frac{1}{2}xy\cdot I_{\{0<y<x<2\}}}{\frac{x^3}{4}\cdot I_{\{0<x<2\}}}=\frac{2y}{x^2}\cdot I_{\{0<y<x<2\}}.$$

Conditional expectation then:

$$g(x)=\mathsf{E}(Y\mid X=x)=\int\limits_{-\infty}^{\infty} y\cdot f_{Y|X}(y\mid x)dy=\int\limits_0^x y\cdot\frac{2y}{x^2}dy\cdot I_{\{0<x<2\}}=$$
$$=\frac{2}{x^2}\cdot\left.\frac{y^3}{3}\right|_0^x\cdot I_{\{0<x<2\}}=\boxed{\frac{2}{3}x\cdot I_{\{0<x<2\}}}.$$

(d) By composition of p.d.f.X is always greater than Y. Since $\mathsf{P}\left(X^2>Y^2\right)\equiv\mathsf{P}(X>Y)$:

$$\mathsf{P}\left(X^2>Y^2\right)=\boxed{1}.$$

Explicit calculation of $\mathsf{P}(X>Y)$ is executed in normalization part.

Problem 5

X is a random variable with p.d.f.

$$f(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{1}{4}, & 0 < x \le 1, \\ ax - a, & 1 < x \le 2, \\ \dfrac{1}{4}, & 2 < x \le 3, \\ 0, & x > 3. \end{cases}$$

(a) Find c.d.f.

(b) Find $\mathsf{P}\left(X > \dfrac{3}{2}\right)$.

(c) Find $\mathsf{P}\left(X < \dfrac{5}{2} \,\middle|\, X > 1\right)$.

(d) Find $\mathsf{E}(X)$.

Solution:

(a) Firstly, let's find the missing value of the parameter a. It can be acquired from the normality condition:

$$\int\limits_{-\infty}^{\infty} f(x)dx = 1.$$

Substituting given p.d.f. and considering different integration limits:

$$\int\limits_0^1 \frac{1}{4} \cdot dx + \int\limits_1^2 (ax - a)dx + \int\limits_2^3 \frac{1}{4} \cdot dx = 1.$$

It's clear that the first and the third terms (from direct calculation and from the geometric view of the p.d.f. in fig. 7) both equal to $\dfrac{1}{4}$. It means that parameter a can be found as:

$$a = \frac{1 - \frac{1}{4} - \frac{1}{4}}{\int_1^2 (x-1)dx} = \frac{\frac{1}{2}}{\left.\frac{x^2}{2}\right|_1^2 - \left. x\right|_1^2} = 1.$$

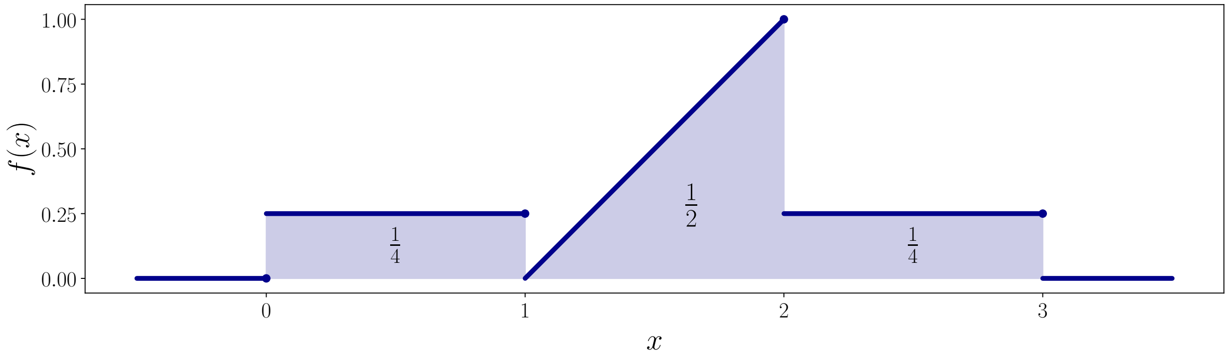

Figure 7: Probability density function $f(x)$.

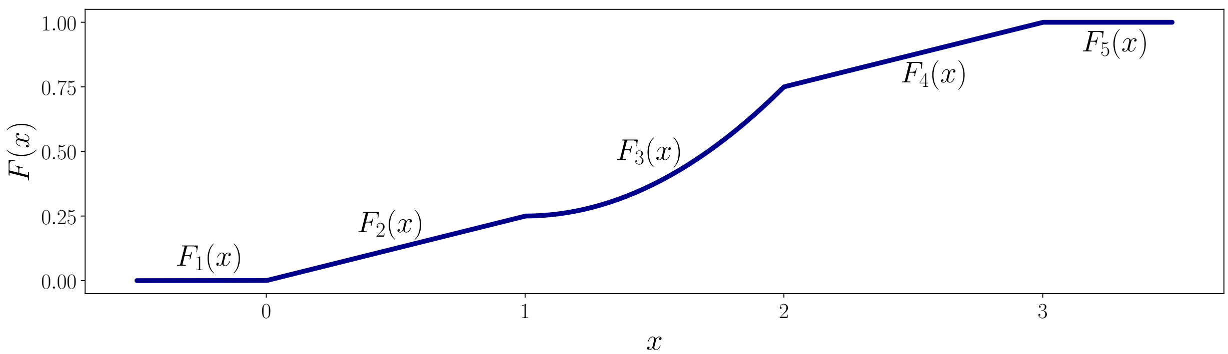

Figure 8: Cumulative distribution function $F(x)$.

Now we are ready to calculate the c.d.f. $F(x)$. As the p.d.f. $f(x)$, it will have 5 different regions. From the property of c.d.f.-s it's clear that the first part for $x \leq 0$ will be zero probability – $F_1(x) = 0$, and the last part for $x > 3$ will be total probability – $F_5(x) = 1$.

Let's calculate the remaining parts by definition of c.d.f.:

$$F(x) = \int_{-\infty}^{x} f(\xi) d\xi.$$

For $x \in (0, 1]$:

$$F_2(x) = \int_{0}^{x} \frac{1}{4} \cdot d\xi = \frac{\xi}{4}\Big|_{0}^{x} = \frac{x}{4}.$$

For $x \in (1, 2]$:

$$F_3(x) = \int_0^1 \frac{1}{4} \cdot d\xi + \int_1^x (\xi - 1) d\xi = F_2(1) + \frac{\xi^2}{2}\Big|_1^x - \xi\Big|_1^x = \frac{x^2}{2} - x + \frac{3}{4}.$$

For $x \in (2, 3]$:

$$F_4(x) = \int_0^1 \frac{1}{4} \cdot d\xi + \int_1^2 (\xi - 1) d\xi + \int_2^x \frac{1}{4} \cdot d\xi = F_3(2) + \frac{\xi}{4}\Big|_2^x = \frac{x+1}{4}.$$

Thus, the c.d.f. has the following view (fig. 8):

$$F(x) = \boxed{\begin{cases} 0, & x \le 0, \\ \dfrac{x}{4}, & 0 < x \le 1, \\ \dfrac{x^2}{2} - x + \dfrac{3}{4}, & 1 < x \le 2, \\ \dfrac{x+1}{4}, & 2 < x \le 3, \\ 1, & x > 3. \end{cases}}$$

(b) Using the supplement property of the c.d.f.:

$$\mathsf{P}\left(X > \frac{3}{2}\right) = 1 - \mathsf{P}\left(X \le \frac{3}{2}\right) = 1 - F\left(\frac{3}{2}\right) = 1 - F_3\left(\frac{3}{2}\right) = \boxed{\frac{5}{8}}.$$

(c) By definition of the conditional probability:

$$\mathsf{P}\left(X < \frac{5}{2} \,\middle|\, X > 1\right) = \frac{\mathsf{P}\left(X < \frac{5}{2} \,\cap\, X > 1\right)}{\mathsf{P}\left(X > 1\right)} = \frac{\mathsf{P}\left(1 < X < \frac{5}{2}\right)}{\mathsf{P}\left(X > 1\right)} =$$

$$= \frac{F\left(\frac{5}{2}\right) - F\left(1\right)}{1 - F\left(1\right)} = \frac{F_4\left(\frac{5}{2}\right) - F_2\left(1\right)}{1 - F_2\left(1\right)} = \frac{\frac{7}{8} - \frac{1}{4}}{1 - \frac{1}{4}} = \boxed{\frac{5}{6}}.$$

(d) Using the definition of the expected value:

$$\mathsf{E}(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^1 x \cdot \frac{1}{4} \cdot dx + \int_1^2 x\left(x - 1\right) dx + \int_2^3 x \cdot \frac{1}{4} \cdot dx =$$

$$= \frac{x^2}{8}\Big|_0^1 + \frac{x^3}{3}\Big|_1^2 - \frac{x^2}{2}\Big|_1^2 + \frac{x^2}{8}\Big|_2^3 = \boxed{\frac{19}{12}}.$$

Problem 6

Let X has uniform distribution $\mathcal{U}(0,1)$. $Y = X^2$.

(a) Find c.d.f. of Y.

(b) Find p.d.f. of Y.

Solution:

(a) Using the definition of the c.d.f. and substituting Y in terms of X:

$$F_Y(y) = \mathsf{P}(Y \leq y) = \mathsf{P}\left(X^2 \leq y\right).$$

Since X^2 on the support $[0,1]$ is a monotone function and takes only positive values:

$$F_Y(y) = \mathsf{P}\left(X^2 \leq y\right) = \mathsf{P}\left(X \leq \sqrt{y}\right) = F_X\left(\sqrt{y}\right).$$

Let's find the c.d.f. of X for $x \in [0,1]$:

$$F_X(x) = \int\limits_{-\infty}^{x} p_X(\xi) d\xi = \int\limits_{0}^{x} 1 \cdot d\xi = x,$$

where p.d.f. of X is $p_X(x) = 1 \cdot I_{\{0 \leq x \leq 1\}}$. It means that c.d.f. of Y is:

$$F_Y(y) = \boxed{\begin{cases} 0, & y < 0, \\ \sqrt{y}, & 0 \leq y \leq 1, \\ 1, & y > 1. \end{cases}}.$$

(b) The p.d.f. of a random variable is the derivative of its c.d.f.:

$$p_Y(y) = \frac{d}{dy} F_Y(y) = \boxed{\frac{1}{2\sqrt{y}} \cdot I_{\{0 \leq y \leq 1\}}}.$$

Another solution would be to use the explicit equation for the substitutions of variables in p.d.f. (which is basically the same we have previously done, but convoluted into one formula):

$$p_Y(y) = \frac{dg^{-1}(y)}{dy} p_X(g^{-1}(y)),$$

where g is the function, which relates Y with X: $Y = g(X)$. In our case g is the square, and g^{-1} is the square root. Thus,

$$p_Y(y) = \frac{d\left(\sqrt{y}\right)}{dy} p_X\left(\sqrt{y}\right) = \frac{1}{2\sqrt{y}} \cdot 1 \cdot I_{\left\{0 \leq \sqrt{y} \leq 1\right\}} = \frac{1}{2\sqrt{y}} \cdot I_{\{0 \leq y \leq 1\}}.$$

This way to solve the problem is not preferable because you can easily get confused if the function on the given support is not monotone.

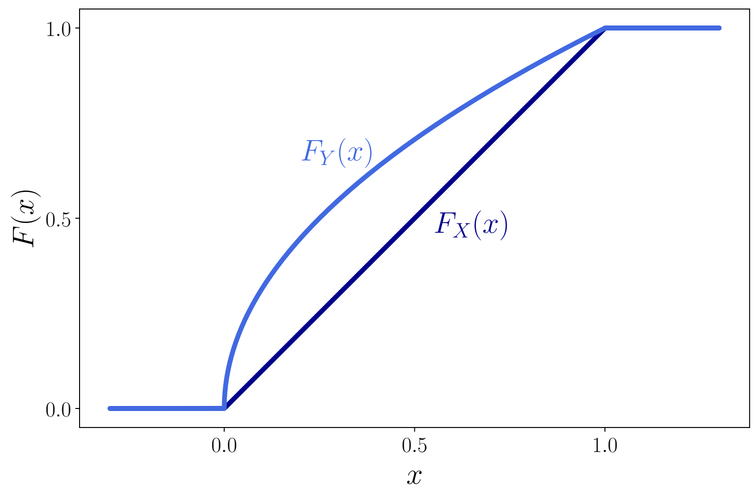

Figure 9: Cumulative distribution functions of $X \sim \mathcal{U}(0, 1)$ and $Y = X^2$.

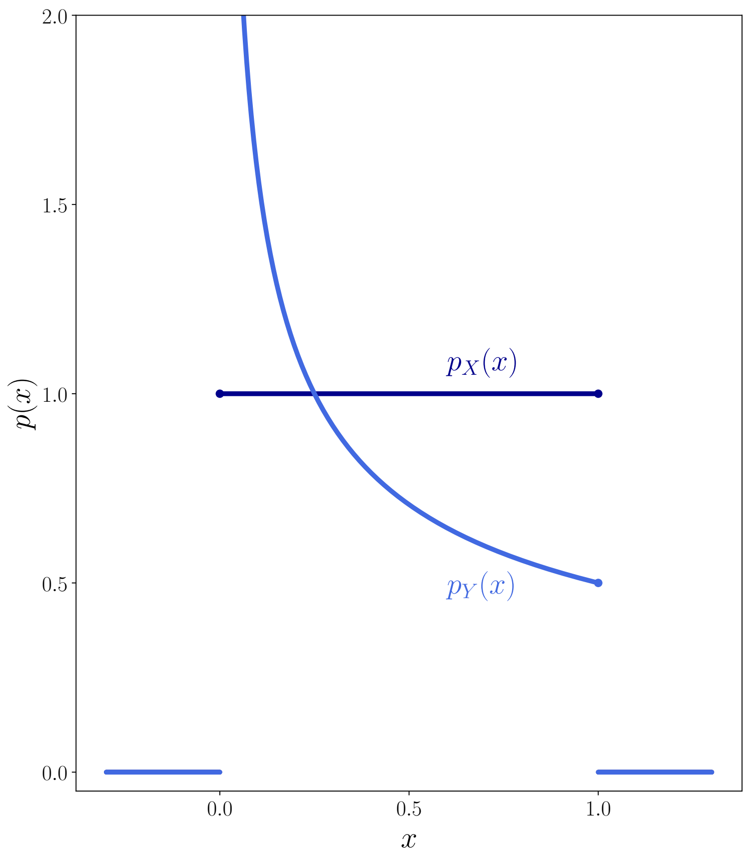

Figure 10: Probability density functions of $X \sim \mathcal{U}(0, 1)$ and $Y = X^2$.

Problem 7

Let X has uniform distribution $\mathcal{U}(-1, 1)$. $Y = X^2$.

(a) Find c.d.f. of Y.

(b) Find p.d.f. of Y.

Solution:

(a) X^2 in this case is a non-monotonous function and c.d.f.s have following equivalence:

$$F_Y(y) = \mathsf{P}(Y \leq y) = \mathsf{P}\left(X^2 \leq y\right) = \mathsf{P}\left(-\sqrt{y} \leq X \leq \sqrt{y}\right) = F_X\left(\sqrt{y}\right) - F_X\left(-\sqrt{y}\right).$$

Let's find the c.d.f. of X for $x \in [-1, 1]$:

$$F_X(x) = \int\limits_{-\infty}^{x} p_X(\xi) d\xi = \int\limits_{-1}^{x} \frac{1}{2} \cdot d\xi = \frac{x+1}{2},$$

where p.d.f. of X is $p_X(x) = \dfrac{1}{2} \cdot I_{\{-1 \leq x \leq 1\}}$. It means that c.d.f. of Y is:

$$F_Y(y) = \begin{cases} 0 - 0, & y < 0, \\ \dfrac{\sqrt{y}+1}{2} - \dfrac{-\sqrt{y}+1}{2}, & 0 \leq y \leq 1, \\ 1 - 0, & y > 1. \end{cases} = \boxed{\begin{cases} 0, & y < 0, \\ \sqrt{y}, & 0 \leq y \leq 1, \\ 1, & y > 1. \end{cases}},$$

since the interval $x \in [-1, 1]$ is translated into interval $y \in [0, 1]$.

(b) As in **Problem 6**, since we have identical $F_Y(y)$, the p.d.f. of Y is:

$$p_Y(y) = \frac{d}{dy} F_Y(y) = \boxed{\frac{1}{2\sqrt{y}} \cdot I_{\{0 \leq y \leq 1\}}}.$$

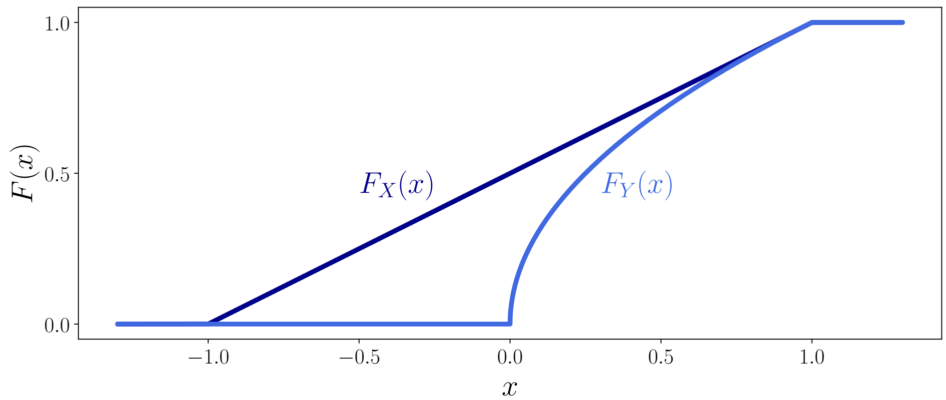

Figure 11: Cumulative distribution functions of $X \sim \mathcal{U}(-1, 1)$ and $Y = X^2$.

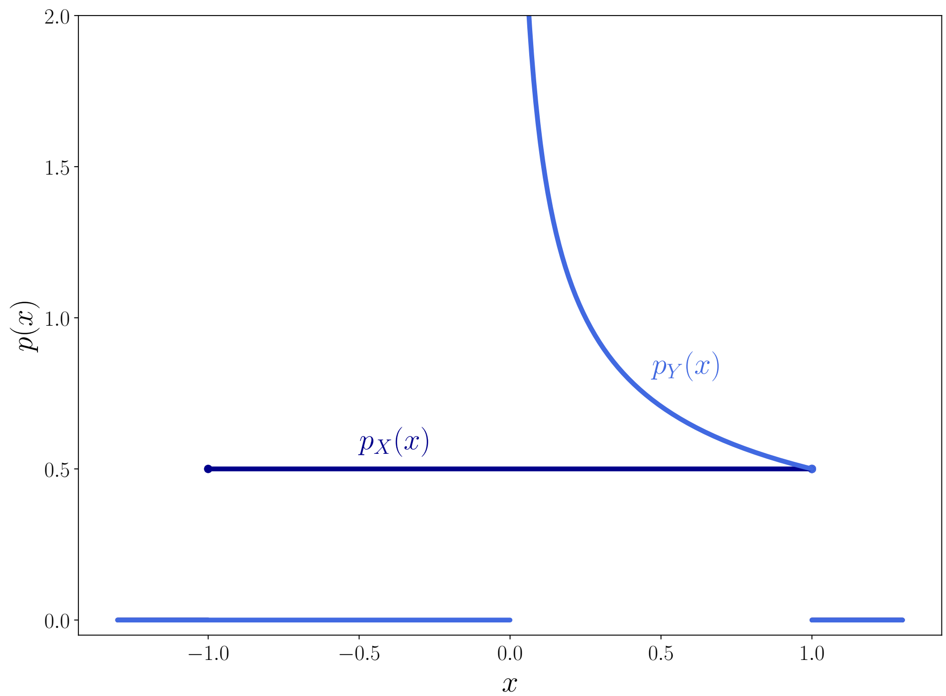

Figure 12: Probability density functions of $X \sim \mathcal{U}(-1, 1)$ and $Y = X^2$.

Problem 8

Let X, Y are independent random variables, distributed uniformly on $[0, 2]$.
Let $W = \max\{X, Y\}$, $L = \min\{X, Y\}$.

(a) Find c.d.f. of W.

(b) Find p.d.f. of W.

(c) Find $\mathsf{E}(W)$, $\mathsf{P}(W < \mathsf{E}(W))$.

(d) Find $\mathsf{E}(L)$.

Solution:

(a) Since W is the maximal value among X and Y, we can make a conclusion, that when W is less than some chosen number t, X and Y are also less than t (simultaneously):

$$\mathsf{P}(W \le t) = \mathsf{P}(X \le t \ \cap \ Y \le t) \overset{\text{ind}}{=} \mathsf{P}(X \le t) \cdot \mathsf{P}(Y \le t).$$

Probability of random variable being less than a chosen value t is the c.d.f. value in t. C.d.f.-s of X and Y are identical and have the following form (for $t \in [0, 2]$):

$$F_X(t) = F_Y(t) = \int\limits_{-\infty}^{t} p_X(x)dx = \int\limits_{0}^{t} \frac{1}{2} \cdot dx = \frac{t}{2},$$

where p.d.f.-s of X and Y are $p_X(x) = p_Y(x) = \dfrac{1}{2} \cdot I_{\{0 \le x \le 2\}}$.

The c.d.f. of W then:

$$F_W(t) = \mathsf{P}(W \le t) = F_X(t) \cdot F_Y(t) = \begin{cases} 0 \cdot 0, & t < 0, \\ \dfrac{t}{2} \cdot \dfrac{t}{2}, & 0 \le t \le 2, \\ 1 \cdot 1, & t > 2, \end{cases} = \boxed{\begin{cases} 0, & t < 0, \\ \dfrac{t^2}{4}, & 0 \le t \le 2, \\ 1, & t > 2. \end{cases}}$$

(b) The p.d.f. of a random variable is the derivative of its c.d.f.:

$$p_W(t) = \frac{d}{dt} F_W(t) = \boxed{\frac{t}{2} \cdot I_{\{0 \le t \le 2\}}}.$$

(c) Using the definition of the expected value:

$$\mathsf{E}(W) = \int_{-\infty}^{\infty} x p_W(x) dx = \int_0^2 x \cdot \frac{x}{2} dx = \frac{x^3}{6}\bigg|_0^2 = \boxed{\frac{4}{3}}.$$

Using the definition of c.d.f.:

$$\mathsf{P}(W < \mathsf{E}(W)) = F_W(\mathsf{E}(W)) = F_W\left(\frac{4}{3}\right) = \boxed{\frac{4}{9}}.$$

(d) Firstly, let's find the c.d.f. of L. Applying same reasoning as for W, in this case we find that if L is greater than t, X and Y are also greater than t. Using survival functions $(1 - \text{c.d.f.})$:

$$\begin{aligned}
F_L(t) &= \mathsf{P}(L \le t) = 1 - \mathsf{P}(L > t) = 1 - \mathsf{P}(X > t \cap Y > t) \overset{\text{ind}}{=} \\
&\overset{\text{ind}}{=} 1 - \mathsf{P}(X > t) \cdot \mathsf{P}(Y > t) = 1 - (1 - \mathsf{P}(X \le t)) \cdot (1 - \mathsf{P}(Y \le t)) = \\
&= 1 - (1 - F_X(t)) \cdot (1 - F_Y(t)) = \begin{cases} 1 - (1-0)\cdot(1-0), & t < 0, \\ 1 - \left(1 - \frac{t}{2}\right)\cdot\left(1 - \frac{t}{2}\right), & 0 \le t \le 2, \\ 1 - (1-1)\cdot(1-1), & t > 2, \end{cases} = \\
&= \begin{cases} 1 - 1, & t < 0, \\ 1 - \left(1 - t + \frac{t^2}{4}\right), & 0 \le t \le 2, \\ 1 - 0, & t > 2, \end{cases} = \begin{cases} 0, & t < 0, \\ t - \frac{t^2}{4}, & 0 \le t \le 2, \\ 1, & t > 2. \end{cases}
\end{aligned}$$

Now the p.d.f. from c.d.f.:

$$p_L(t) = \frac{d}{dt} F_L(t) = \left(1 - \frac{t}{2}\right) \cdot I_{\{0 \le t \le 2\}}.$$

And the expected value of L is:

$$\mathsf{E}(L) = \int_{-\infty}^{\infty} x p_L(x) dx = \int_0^2 x\left(1 - \frac{x}{2}\right) dx = \frac{x^2}{2}\bigg|_0^2 - \frac{x^3}{6}\bigg|_0^2 = \boxed{\frac{2}{3}}.$$

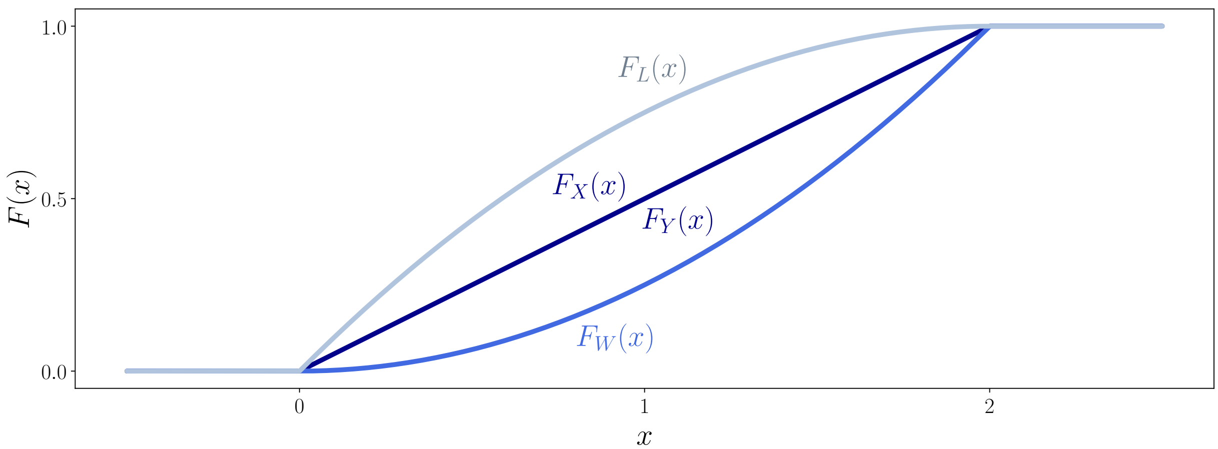

Figure 13: Cumulative distribution functions of X, Y, $W = \max(X, Y)$ and $L = \min(X, Y)$.

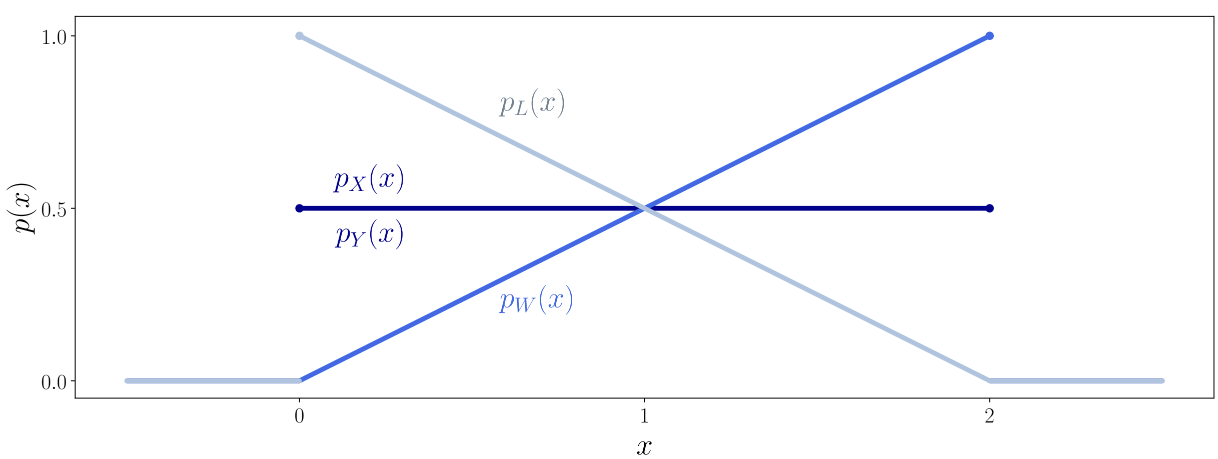

Figure 14: Probability density functions of X, Y, $W = \max(X, Y)$ and $L = \min(X, Y)$.

Problem 9

Let X be a random variable with uniform distribution on the interval $[0, \pi]$. Find p.d.f. of random variables:

(a) $Y = \sin X$,

(b) $Z = X^3$.

Solution:

P.d.f. f_X and c.d.f. F_X of X are following:

$$f_X(x) = \frac{1}{\pi} \cdot I_{\{0 \le x \le \pi\}}, \qquad F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{\pi}, & 0 \le x \le \pi, \\ 1, & x > \pi. \end{cases}$$

(a) Let's find c.d.f. of Y:

$$F_Y(y) = \mathsf{P}(Y \le y) = \mathsf{P}(\sin X \le y).$$

Since $\sin X$ is not a monotonous function, we should find a range of X, which supports $\mathsf{P}(\sin X \le y)$ within the interval $[0, \pi]$. This range is illustrated in the fig. 15: From

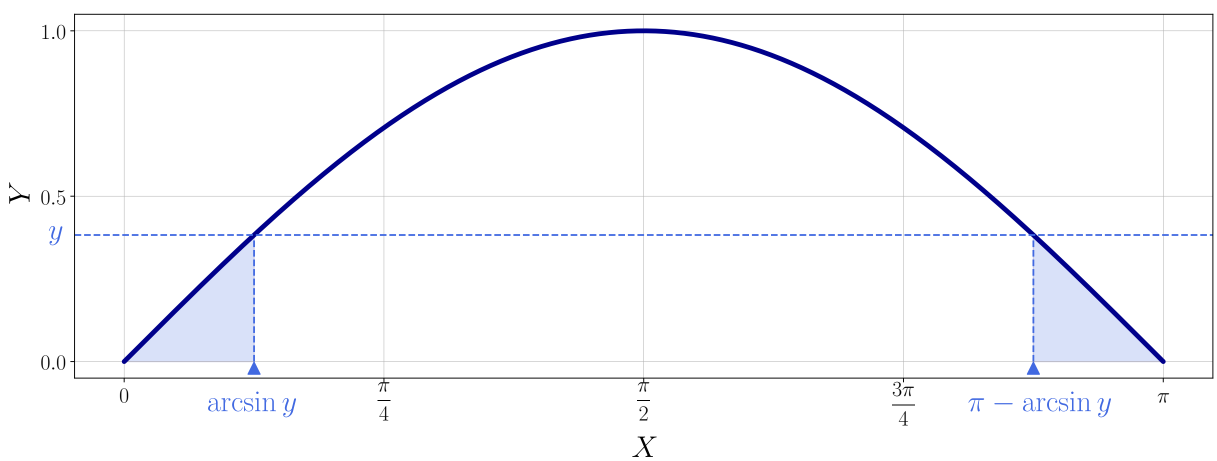

Figure 15: Conversion of probability from X to Y.

the figure it's clear that if $y < 0$ the probability is 0, if $y > 1$ the probability is 1, and if $0 \le y \le 1$:

$$\mathsf{P}(\sin X \le y) = \mathsf{P}(0 \le X \le \arcsin y) + \mathsf{P}(\pi - \arcsin y \le X \le \pi) =$$
$$= F_X(\arcsin y) - F_X(0) + F_X(\pi) - F_X(\pi - \arcsin y) = \frac{2}{\pi} \arcsin y.$$

It means that c.d.f. of Y is:

$$F_Y(y) = \begin{cases} 0, & y < 0, \\ \dfrac{2}{\pi} \arcsin y, & 0 \le y \le 1, \\ 1, & y > 1. \end{cases}$$

P.d.f. $f_Y(y)$ is a derivative of c.d.f. $F_Y(y)$:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \boxed{\frac{2}{\pi} \frac{1}{\sqrt{1-y^2}} \cdot I_{\{0 \le y < 1\}}}.$$

Here we have to exclude 1 from p.d.f. due to discontinuity.

(b) Since $Z = X^3$ is a monotonous function on the interval $[0, \pi]$, probabilities are converted straightforwardly:

$$F_Z(z) = \mathsf{P}(Z \le z) = \mathsf{P}\left(X^3 \le z\right) = \mathsf{P}\left(X \le \sqrt[3]{z}\right) = F_X\left(\sqrt[3]{z}\right),$$

where support on Z is stretched out from $[0, 1]$ to $[0, \pi^3]$. The c.d.f. then:

$$F_Z(z) = \begin{cases} 0, & z < 0, \\ \dfrac{\sqrt[3]{z}}{\pi}, & 0 \le z \le \pi^3, \\ 1, & z > \pi^3. \end{cases}$$

P.d.f. $f_Y(y)$ is a derivative of c.d.f. $F_Y(y)$:

$$f_Z(z) = \frac{d}{dz} F_Z(z) = \boxed{\frac{1}{3\pi z^{2/3}} \cdot I_{\{0 < z \le \pi^3\}}}.$$

Here we have to exclude 0 from p.d.f. due to discontinuity.

Illustrations of c.d.f.s and p.d.f.s of variables X, Y, Z are in the fig. 16 and 17 respectively. Note the slopes of c.d.f.s and how they are connected to p.d.f. values.

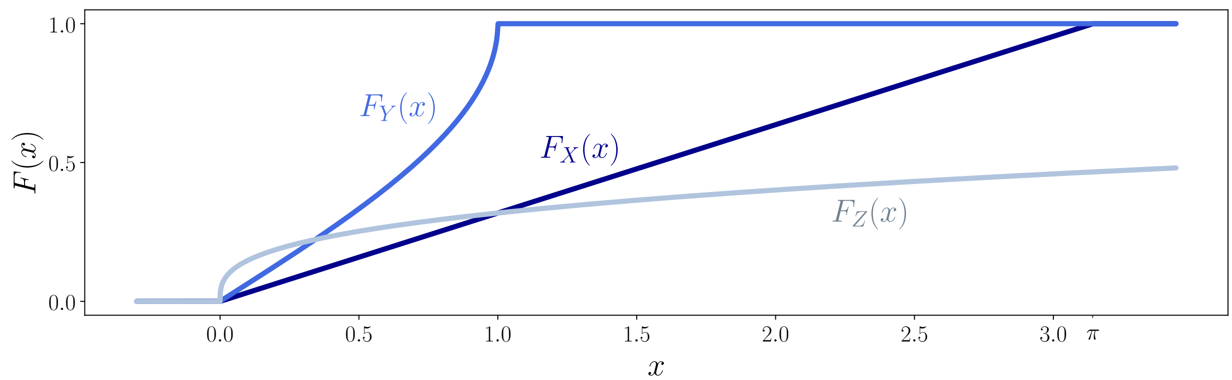

Figure 16: Cumulative distribution functions of $X \sim \mathcal{U}(0, \pi)$, $Y = \sin X$ and $Z = X^3$.

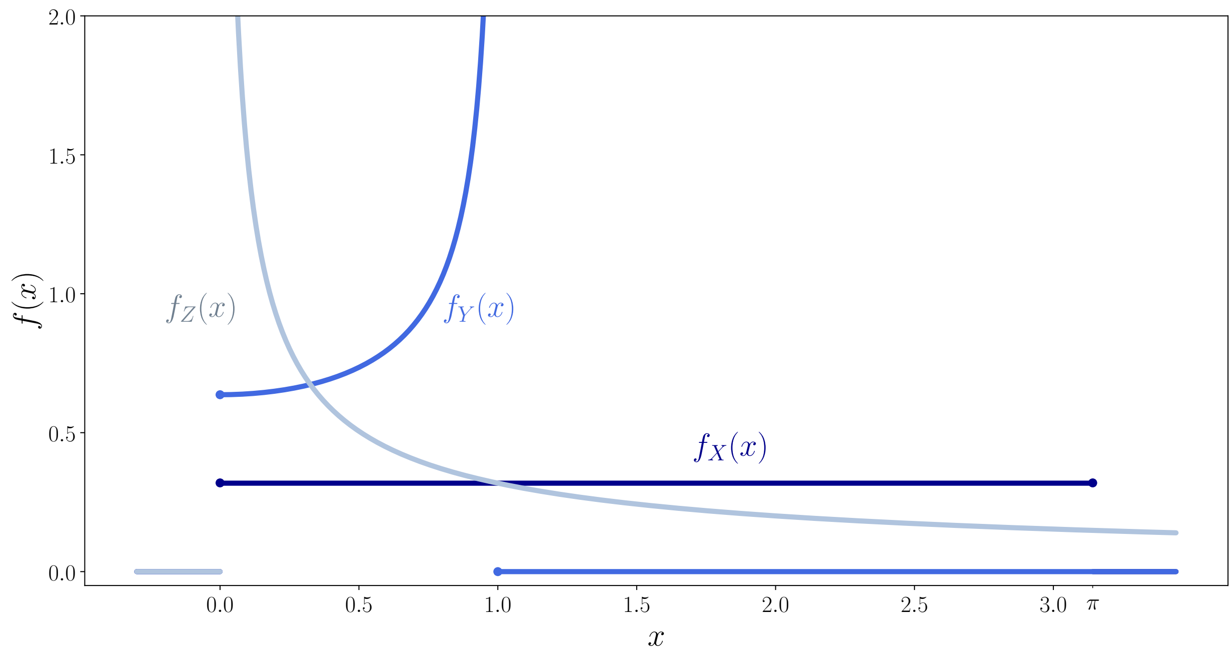

Figure 17: Probability density functions of $X \sim \mathcal{U}(0, \pi)$, $Y = \sin X$ and $Z = X^3$.

Problem 10

Suppose there are three assets with returns X_1, X_2, and X_3. It is known that the returns are uncorrelated and their means and standard deviations are:

$$\mu_1 = 0.10, \mu_2 = 0.05, \mu_3 = 0.02,$$

$$\sigma_1 = 0.40, \sigma_2 = 0.20, \sigma_3 = 0.05.$$

Find the "optimal" portfolio with mean $\mu = 0.06$ ("optimal" means smallest variance).

Solution:

The portfolio is a linear combination of three assets:

$$X = k_1 X_1 + k_2 X_2 + k_3 X_3$$

with constraint on a mean

$$\mathsf{E}(X) = \mu$$

and minimization of a variance

$$\mathsf{V}(X) \to \min .$$

Since assets are uncorrelated, the last two expressions have the following view:

$$\begin{aligned} k_1\mu_1 + k_2\mu_2 + k_3\mu_3 &= \mu, \\ k_1^2\sigma_1^2 + k_2^2\sigma_2^2 + k_3^2\sigma_3^2 &\to \min . \end{aligned}$$

Let's use Lagrange method with constraints to find optimal coefficients. The Lagrangian: $\mathcal{L}$

$$\mathcal{L} = k_1^2\sigma_1^2 + k_2^2\sigma_2^2 + k_3^2\sigma_3^2 - \lambda(k_1\mu_1 + k_2\mu_2 + k_3\mu_3 - \mu).$$

According to the FOC of extrema:

$$\begin{cases} \left.\dfrac{\partial \mathcal{L}}{\partial k_1}\right|_{(k_1,k_2,k_3,\lambda)=(k_1^\star,k_2^\star,k_3^\star,\lambda^\star)} = 0, \\ \left.\dfrac{\partial \mathcal{L}}{\partial k_2}\right|_{(k_1,k_2,k_3,\lambda)=(k_1^\star,k_2^\star,k_3^\star,\lambda^\star)} = 0, \\ \left.\dfrac{\partial \mathcal{L}}{\partial k_3}\right|_{(k_1,k_2,k_3,\lambda)=(k_1^\star,k_2^\star,k_3^\star,\lambda^\star)} = 0, \\ \left.\dfrac{\partial \mathcal{L}}{\partial \lambda}\right|_{(k_1,k_2,k_3,\lambda)=(k_1^\star,k_2^\star,k_3^\star,\lambda^\star)} = 0, \end{cases}$$

we have

$$\begin{cases} 2\sigma_1^2 k_1^\star - \mu_1\lambda^\star = 0, \\ 2\sigma_2^2 k_2^\star - \mu_2\lambda^\star = 0, \\ 2\sigma_3^2 k_3^\star - \mu_3\lambda^\star = 0, \\ \mu_1 k_1^\star + \mu_2 k_2^\star + \mu_3 k_3^\star = \mu. \end{cases}$$

Expressing $k_i^\star$ from the 1st three equations and substituting them into the 4th one:

$$\begin{cases} k_1^\star = \dfrac{\mu_1}{2\sigma_1^2}\lambda^\star, \\ k_2^\star = \dfrac{\mu_2}{2\sigma_2^2}\lambda^\star, \\ k_3^\star = \dfrac{\mu_3}{2\sigma_3^2}\lambda^\star, \\ \mu_1 k_1^\star + \mu_2 k_2^\star + \mu_3 k_3^\star = \mu. \end{cases} \implies \lambda^\star = \frac{2\mu}{\dfrac{\mu_1^2}{\sigma_1^2} + \dfrac{\mu_2^2}{\sigma_2^2} + \dfrac{\mu_3^2}{\sigma_3^2}} = \frac{24}{57}.$$

Thus

$$\begin{cases} k_1^\star = \dfrac{\mu_1}{\sigma_1^2} \cdot \dfrac{\mu}{\dfrac{\mu_1^2}{\sigma_1^2} + \dfrac{\mu_2^2}{\sigma_2^2} + \dfrac{\mu_3^2}{\sigma_3^2}} = \dfrac{5}{38}, \\ k_2^\star = \dfrac{\mu_2}{\sigma_2^2} \cdot \dfrac{\mu}{\dfrac{\mu_1^2}{\sigma_1^2} + \dfrac{\mu_2^2}{\sigma_2^2} + \dfrac{\mu_3^2}{\sigma_3^2}} = \dfrac{5}{19}, \\ k_3^\star = \dfrac{\mu_3}{\sigma_3^2} \cdot \dfrac{\mu}{\dfrac{\mu_1^2}{\sigma_1^2} + \dfrac{\mu_2^2}{\sigma_2^2} + \dfrac{\mu_3^2}{\sigma_3^2}} = \dfrac{32}{19}. \end{cases}$$

The optimal portfolio is:

$$\boxed{X = \frac{5}{38}X_1 + \frac{5}{19}X_2 + \frac{32}{19}X_3}.$$

The found extremum is a minimum, since the second differential of a Lagrangian is always positive:

$$d^2\mathcal{L} = 2\sigma_1^2 dk_1^2 + 2\sigma_2^2 dk_2^2 + 2\sigma_3^2 dk_3^2 > 0.$$