# Geometric probability. Conditional probability

## Probability theory

Anton Afanasev

Higher School of Economics

DSBA 221

September 15, 2023

# Seminar Overview

1. Quiz
2. Geometric probability
   - Problems
   - Properties
3. Conditional probability
   - Definition
   - Independence
   - Bayes' theorem

# Quiz

Remember BlackJack?
You're given 2 random cards from the same deck of 52.

What's the probability of getting

1. 21 pts,
2. 20 pts,
3. 12 pts,
4. 10 pts?

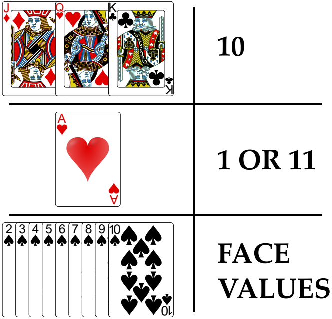



Figure: Card values.

*Note*: the value of an ace maximizes the score if it does not exceed 21.

# Problem 1

Two points, $X$ and $Y$, are randomly chosen on an interval $OA = [0, 1]$. Find the probability of each of the following events:

1. A distance between $X$ and $O$ is less than $\frac{1}{10}$.
2. A distance between $X$ and $O$ is between 0.7 and 0.705.
3. A distance between $X$ and $O$ is equal to 0.7
4. A distance between $X$ and $Y$ is less than 0.5
5. A distance between $X$ and $Y$ is equal to $\frac{1}{3}$.
6. Length of $XY$ is less than the distance between $O$ and the closest point to it.

# Problem 2

Consider a round shooting target with a radius of $R$. Someone is shooting at it with bullets of radius $B$. Find the probability that a hole made by the shot entirely lies within an interior circle with a radius of $r$. Assume $R > r > B$.

# Problem 3

A stick of length $L$ is broken in two places. The break points are independent of each other and are chosen at random (uniformly) on the stick. What is the probability that a triangle can be formed using these three pieces of stick?

# Properties of geometric probability

- Notion of the cardinality of an event $|A|$ from classical probability expands to the measure of an event $\mu(A)$:

$$\mathsf{P}_{\text{class}}(A) = \frac{|A|}{|\Omega|} \qquad \Longrightarrow \qquad \mathsf{P}_{\text{geom}}(A) = \frac{\mu(A)}{\mu(\Omega)}.$$

- Impossible $\neq$ Improbable:

$$\mathsf{P}_{\text{geom}}(\omega) = \mu(\omega) = 0.$$

- Comparison with a classical probability:

1. $\mathsf{P}_{\text{class}}(A) \in [0,1]$,
2. $\mathsf{P}_{\text{class}}(A) = 1 \quad \Longleftrightarrow \quad A = \Omega$,
3. $\mathsf{P}_{\text{class}}\left(\overline{A}\right) = 1 - \mathsf{P}_{\text{class}}(A)$,
4. closed under $\bigcap$ and $\bigcup$.

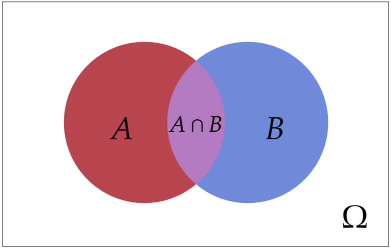
1. $\mathsf{P}_{\text{geom}}(A) \in [0,1]$,
2. $\mathsf{P}_{\text{geom}}(A) = 1 \quad \Longleftarrow \quad A = \Omega$,
3. $\mathsf{P}_{\text{geom}}\left(\overline{A}\right) = 1 - \mathsf{P}_{\text{geom}}(A)$,
4. closed under $\bigcap$ and $\bigcup$.

# Problem 4

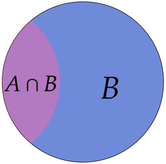
There are two children in a family. It is known that at least one of them is a boy. Find the probability that the other child is also a boy.

# Conditional probability

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} = \frac{\mu(A \cap B)/\mu(\Omega)}{\mu(B)/\mu(\Omega)} = \frac{\mu(A \cap B)}{\mu(B)}.$$




(a) Original sample space $\Omega$




(b) Narrowed sample space $\Omega \to B$

Figure: Geometric interpretation of conditional probability.

# Independence

## Definition

Two events $A$ and $B$ are independent iff:

$$\mathsf{P}(A \cap B) = \mathsf{P}(A) \cdot \mathsf{P}(B).$$

- If $A$ and $B$ are independent then:

$$\mathsf{P}(A \mid B) = \frac{\mathsf{P}(A \cap B)}{\mathsf{P}(B)} = \frac{\mathsf{P}(A) \cdot \mathsf{P}(B)}{\mathsf{P}(B)} = \mathsf{P}(A).$$

- Don't confuse independence with mutual exclusion:

$$A \cap B = \varnothing \qquad \Longrightarrow \qquad \mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B).$$

# Pairwise and collective independence

**Definition**

Events $A_1, \ldots, A_n$ are pairwise independent iff:

$$\forall i \neq j \in \overline{1, n} \qquad \mathsf{P}(A_i \cap A_j) = \mathsf{P}(A_i) \cdot \mathsf{P}(A_j).$$

**Definition**

Events $A_1, \ldots, A_n$ are collectively (mutually) independent iff:

$$\forall I \subset \overline{1, n} \qquad \mathsf{P}\left(\bigcap_{i \in I} A_i\right) = \prod_{i \in I} \mathsf{P}(A_i).$$

- For three events $A$, $B$ and $C$:

$$\begin{array}{lllll} \text{pairwise:} & \mathsf{P}(A \mid B) = \mathsf{P}(A), & \mathsf{P}(C \mid A) = \mathsf{P}(C), & \ldots \\ \text{collective:} & \mathsf{P}(A \mid B, C) = \mathsf{P}(A), & \mathsf{P}(C \mid A, B) = \mathsf{P}(C), & \ldots \end{array}$$

# Problem 5

Two coins are tossed. Let's denote events $A, B$ and $C$ the following way:

$$A = \{1^{\text{st}} \text{ coin is heads}\},$$
$$B = \{2^{\text{nd}} \text{ coin is heads}\},$$
$$C = \{\text{only one coin is heads}\}.$$

Are these events collectively independent? Are they pairwise independent? Will the situation change if the coin is not fair?

# Total probability

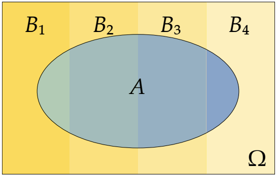



Figure: Set $A$ in terms of intersections with sets $B_i$.

- Total probability of $A$:

$$\mathsf{P}(A) = \sum_{i=1}^{n} \mathsf{P}(A \cap B_i) = \sum_{i=1}^{n} \mathsf{P}(A \mid B_i) \cdot \mathsf{P}(B_i).$$

- If you care about specific $B_i$:

$$\mathsf{P}(A) = \mathsf{P}(A \mid B_i) \cdot \mathsf{P}(B_i) + \mathsf{P}(A \mid \overline{B}_i) \cdot \mathsf{P}(\overline{B}_i).$$

- Set of $\{B_1, \ldots, B_n\}$ should be collectively exhaustive and mutually exclusive.

# Bayes' theorem

Using definition of conditional probability twice:

$$\mathsf{P}(A \cap B) = \mathsf{P}(A \mid B) \cdot \mathsf{P}(B) = \mathsf{P}(B \mid A) \cdot \mathsf{P}(A).$$

**Theorem (Bayes')**

$$\mathsf{P}(B \mid A) = \frac{\mathsf{P}(A \mid B) \cdot \mathsf{P}(B)}{\mathsf{P}(A)}$$

- $\mathsf{P}(B)$ – *prior*, the initial degree of belief in $B$.
- $\mathsf{P}(B \mid A)$ – *posterior*, the degree of belief after incorporating news that $A$ is true.
- When there are several priors:

$$\mathsf{P}(B_i \mid A) = \frac{\mathsf{P}(A \mid B_i) \cdot \mathsf{P}(B_i)}{\sum\limits_{j=1}^{n} \mathsf{P}(A \mid B_j) \cdot \mathsf{P}(B_j)}.$$

# Problem 6

There are three cards.

- The letter $\mathbb{A}$ is written on both sides of the $1^{\text{st}}$ card.
- The letter $\mathbb{A}$ is written on both sides of the $2^{\text{nd}}$ card.
- Letters $\mathbb{A}$ and $\mathbb{B}$ are written on different sides of the $3^{\text{rd}}$ card.

A random card has been put on the table in such a way that the letter $\mathbb{A}$ is visible. What is the probability that the letter $\mathbb{A}$ is written on the other side of the card?

# Problem 7

A system consists of two parallel elements and is operational if at least one of them is working. At a random time, the $1^{st}$ element is out of order with a probability of 0.1, and the $2^{nd}$ element is out of order with a probability of 0.2. Someone has informed us that the system is currently operational. What is the probability that the $2^{nd}$ element is out of order?

That's all Folks