ANOVA confidence intervals

Statistics

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Seminar Overview

1. Quiz
2. Individual confidence intervals
   - One-way ANOVA
   - Two-way ANOVA
3. Simultaneous confidence intervals
   - Family-wise error rate
   - Scheffé's method

Quiz

Three random samples were taken from normal distributions with same standard deviations.

Sample 1	10	20	
Sample 2	5	15	25
Sample 3	8	32	

1. Complete ANOVA table.
2. Test $H_0 : \mu_1 = \mu_2 = \mu_2$ against H_1 : not H_0 at 5% significance level.

Problem 1

Independent random samples of six assistant professors, fours associate professors, and five full professors were asked to estimate the amount of time outside the classroom spent on teaching responsibilities in the last week. Results, in hours, are shown in the accompanying table.

ASSISTANT	ASSOCIATE	FULL
7	15	11
12	12	7
11	15	6
15	8	9
9		7
14		

1. Set out the analysis of variance table.
2. Test the null hypothesis that the three population mean times are equal.

Problem 2

The driver of a diesel-powered automobile decided to test the quality of three types of diesel fuel sold in the area based on the mpg.

Brand A	38.7	39.2	40.1	38.9	
Brand B	41.9	42.3	41.3		
Brand C	40.8	41.2	39.5	38.9	40.3

1. Test the null hypothesis that three means are equal using the following data. Make the usual assumptions and take $\alpha = 0.05$.
2. Calculate individual 95% confidence intervals for the each of 3 differences in the 3 types.

One-way ANOVA model

- Model of one-way ANOVA independent observations:

$$X_{ij} = \mu + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \mathcal{N}\left(0, \sigma^2\right).$$

- Variation, created by ε_{ij} (within-group, residual, error):

$$W = RSS = SSE = \sum_{j=1}^{k} \sum_{i=1}^{n_j} \left(X_{ij} - \overline{X}_{\cdot j}\right)^2$$

with $n - k$ degrees of freedom.

- Point estimate of σ^2:

$$\widehat{\sigma^2} = \frac{W}{n-k} = MSE.$$

One-way ANOVA individual confidence intervals

- $(1-\alpha)\%$ confidence interval for $\mu_j = \mu + \beta_j$:

$$(\mu_j)_{1-\alpha} \in \overline{X}_{\cdot j} \pm t_{n-k;\ \alpha/2} \cdot \sqrt{\frac{MSE}{n_j}}.$$

- $(1-\alpha)\%$ confidence interval for difference $\mu_A - \mu_B$:

$$(\mu_A - \mu_B)_{1-\alpha} \in \overline{X}_{\cdot A} - \overline{X}_{\cdot B} \pm t_{n-k;\ \alpha/2} \cdot \sqrt{MSE\left(\frac{1}{n_A} + \frac{1}{n_B}\right)}.$$

Two-way ANOVA model

- Model of two-way ANOVA independent observations:

$$X_{ij} = \mu + \gamma_i + \beta_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim \mathcal{N}\left(0, \sigma^2\right).$$

- Variation, created by ε_{ij} (residual, error):

$$RSS = SSE = \sum_{i=1}^{r} \sum_{j=1}^{c} \left(X_{ij} - \overline{X}_{i\cdot} - \overline{X}_{\cdot j} + \overline{X}\right)^2$$

  with $(r-1)(c-1)$ degrees of freedom.

- Point estimate of σ^2:

$$\widehat{\sigma^2} = \frac{RSS}{(r-1)(c-1)} = MSE.$$

Two-way ANOVA individual confidence intervals

- $(1-\alpha)\%$ confidence intervals for $\mu_i = \mu + \gamma_i$ and $\mu_j = \mu + \beta_j$:

$$(\mu_i)_{1-\alpha} \in \overline{X}_{i\cdot} \pm t_{(r-1)(c-1);\ \alpha/2} \cdot \sqrt{\frac{MSE}{c}},$$

$$(\mu_j)_{1-\alpha} \in \overline{X}_{\cdot j} \pm t_{(r-1)(c-1);\ \alpha/2} \cdot \sqrt{\frac{MSE}{r}}.$$

- $(1-\alpha)\%$ confidence interval for differences:

$$(\mu_1 - \mu_2)_{1-\alpha} \in \overline{X}_{1\cdot} - \overline{X}_{2\cdot} \pm t_{(r-1)(c-1);\ \alpha/2} \cdot \sqrt{\frac{2\ MSE}{c}},$$

$$(\mu_A - \mu_B)_{1-\alpha} \in \overline{X}_{\cdot A} - \overline{X}_{\cdot B} \pm t_{(r-1)(c-1);\ \alpha/2} \cdot \sqrt{\frac{2\ MSE}{r}}.$$

Problem 3

An instructor in an economic class is considering three different texts. She is also considering three types of examinations – multiple choice, essays, and a mix of multiple choice and essays. During the year she teaches nine sections of this course, and randomly assigns a text-examination type combination to each section. At the end of the course, she obtained students evaluation for each section. These ratings are shown in the accompanying table.

	A	B	C
Multiple choice	4.8	5.3	4.9
Essays	4.6	5.0	4.3
Mix	4.6	5.1	4.8

1. Set out the analysis of variance table.

Problem 3

2. Test the null hypothesis of equality of population ratings for the three texts.
3. Test the null hypothesis of equality of population ratings for the three examination types.
4. Find single 95% confidence intervals for $\mu_A - \mu_B$, $\mu_A - \mu_C$, $\mu_B - \mu_C$.
5. Find simultaneous 95% confidence intervals for $\mu_A - \mu_B$, $\mu_A - \mu_C$, $\mu_B - \mu_C$.

Family-wise error rate

- Family of parameters $\{\theta_1, \theta_2, \dots, \theta_k\}$ with individual confidence intervals:

$$\begin{aligned} \mathsf{P}(L_1 \leq \theta_1 \leq U_1) &= 0.95, \\ \mathsf{P}(L_2 \leq \theta_2 \leq U_2) &= 0.95, \\ &\dots \\ \mathsf{P}(L_k \leq \theta_k \leq U_k) &= 0.95. \end{aligned}$$

- Family-wise error rate (FWER) – probability of making at least one type I error in the family.
- Assuming independence for intervals above:

$$\text{FWER} = 1 - 0.95^k,$$

which is greater than 5% for $k \geq 2$.

Simultaneous confidence intervals

- In order to get FWER $= 5\%$, confidence level of each individual interval should be:
$$1 - \alpha = \sqrt[k]{0.95}.$$
- For $k = 2$ it's 0.975, for $k = 10$ this level is 0.995.
- $(1 - \alpha)\%$ simultaneous confidence intervals for a family of parameters $\{\theta_1, \theta_2, \ldots, \theta_k\}$ is a family of intervals:
$$\{(L_1, U_1), (L_2, U_2), \ldots, (L_k, U_k)\}$$
such that
$$\mathsf{P}(\forall i : L_i \leq \theta_i \leq U_i) = 1 - \alpha.$$
- Simultaneous intervals are wider than individual ones.

Scheffé's method

- One of the most conservative methods of computing a family of simultaneous confidence intervals.
- Family of parameters:

$$\text{all possible combinations of } \sum_{j=1}^{k} d_j \mu_j, \text{ where } \sum_{j=1}^{k} d_j = 0.$$

Those values are called contrasts.

Example

Possible contrasts:

- $\mu_1 - \mu_2 :$ $\quad d_1 = 1, d_2 = -1.$
- $2\mu_1 - \mu_2 - 3\mu_4 + 2\mu_5 :$ $\quad d_1 = 2, d_2 = -1, d_3 = 0, d_4 = -3, d_5 = 2.$
- $\frac{1}{3}\mu_1 + \frac{1}{3}\mu_2 - \frac{2}{3}\mu_3 :$ $\quad d_1 = \frac{1}{3}, d_2 = \frac{1}{3}, d_3 = -\frac{2}{3}.$

Scheffé's method intervals

- One-way ANOVA:

$$\left(\sum_{j=1}^{k} d_j \mu_j\right)_{1-\alpha} \in \sum_{j=1}^{k} d_j \overline{X}_{\cdot j} \pm \sqrt{(k-1) \cdot F_{(k-1);\ (n-k);\ \alpha} \cdot MSE \cdot \sum_{j=1}^{k} \frac{d_j^2}{n_j}}.$$

- Two-way ANOVA:

$$\left(\sum_{i=1}^{r} d_i \mu_i\right)_{1-\alpha} \in \sum_{i=1}^{r} d_i \overline{X}_{i\cdot} \pm \sqrt{(r-1) \cdot F_{(r-1);\ (r-1)(c-1);\ \alpha} \cdot MSE \cdot \sum_{i=1}^{r} \frac{d_i^2}{c}},$$

$$\left(\sum_{j=1}^{c} d_j \mu_j\right)_{1-\alpha} \in \sum_{j=1}^{c} d_j \overline{X}_{\cdot j} \pm \sqrt{(c-1) \cdot F_{(c-1);\ (r-1)(c-1);\ \alpha} \cdot MSE \cdot \sum_{j=1}^{c} \frac{d_j^2}{r}}.$$

Problem 4

The performance of four different traders was monitored over the five different days of the same week. The total profit each trader made over the week measured in thousands of euros was 192 for A, 182 for B, 90 for C and 77 for D. The two-way ANOVA table is as follows

Source	Degrees of Freedom	Sum of Squares	Mean Square	F-value
Traders				1.04
Day				
Error			697	
Total		28477		

1. Complete the table.
2. Is there significant difference between the performances of different traders? What about different days of the week?
3. Construct a 90% confidence interval for the difference in performance between A and D. Would you say there is evidence A is a better trader?
4. Construct simultaneous 90% confidence intervals for the difference between A and D and the difference between B and C.

Problem 5

The performance (yield in kg/sq.m) of six different fertilizers was tested on the four varieties of potatoes (A, B, C, and D). The average yield for each variety was 14.5 for A, 17.5 for B, 14.5 for C and 17 for D. The two-way ANOVA table is as follows

Source	Degrees of Freedom	Sum of Squares	Mean Square	F-value
Fertilizers				
Varieties				
Error			6.775	
Total		274.625		

1. Complete the table.
2. Is there a significant difference between the yields of different varieties? What about different fertilizers? (use 5% significance level)
3. Test hypothesis that average yield of variety B is larger than average yield of variety A (use 5% significance level).
4. What assumptions have you made?

Look at the time!