Bayes' theorem. Bernoulli scheme

Probability theory

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Seminar Overview

1. Quiz
2. Bayes' theorem
   - Recall from last seminar
3. Bernoulli scheme
4. Extra problems

Quiz

For each one of the statements below say whether the statement is true or false explaining your answer. A and B are events such that $0 < \mathsf{P}(A) < 1$ and $0 < \mathsf{P}(B) < 1$.

1. If A and B are independent, then $\mathsf{P}(A) + \mathsf{P}(B) > \mathsf{P}(A \cup B)$.
2. If A and B are independent, then $\mathsf{P}(A) + \mathsf{P}(B) \leq 1$.
3. If $\mathsf{P}(A) < \mathsf{P}(B)$, then $A \subset B$.
4. If $\mathsf{P}(A \mid B) = \mathsf{P}(B \mid A)$ and $\mathsf{P}(A \cap B) > 0$, then $\mathsf{P}(A) = \mathsf{P}(B)$.

Problem 1

Consider a high-risk population where 5% of people have COVID-19. A diagnostic test is correct in 95% of cases if a person has COVID-19 and in 90% of cases if a person does not have COVID-19. If a person tests positive (indicating COVID-19), what is the probability that the person does not have COVID-19?

Problem 2

You have a biased coin for which $\mathsf{P}(H) = p$. You toss the coin 20 times. What is the probability that:

1. you observe first 8 heads and then 12 tails,
2. you observe 8 heads and 12 tails,
3. you observe more than 8 heads and more than 8 tails?

Bernoulli scheme

- The experiment is repeated n times, independently.
- An elementary outcome ω is a set of experiment results y_i:

$$\omega = (y_1, \ldots, y_n).$$

- The probability of success: $p \in [0, 1]$. The probability of failure: $q = 1 - p$.
- Let $y_i = 1$ denote the success in the i^{th} repetition and $y_i = 0$ denote failure. The probability of an elementary outcome then:

$$\mathsf{P}(\omega) = p^{y_1} \cdot q^{1-y_1} \cdot \ldots \cdot p^{y_n} \cdot q^{1-y_n} = p^{\sum\limits_{i=1}^{n} y_i} \cdot q^{n-\sum\limits_{i=1}^{n} y_i}.$$

- The probability to get $k = \sum\limits_{i=1}^{n} y_i$ successes:

$$\mathsf{P}(k \text{ successes}) = C_n^k p^k q^{n-k}.$$

Problem 3

Suppose a good password must consist of two lowercase letters (a to z), followed by one capital letter (A to Z), followed by four digits. For example, *"ejT3018"* is a good password.

1. Find the total number of good passwords.
2. A hacker wrote a program that randomly generated 108 good passwords (one password could be generated more than once). What is the probability that at least one of the generated passwords matches the password of a particular user?
3. Answer the question 2 assuming that the program generated 108 distinct passwords.

Problem 4

The student has learned 20 out of 25 exam questions before the exam. She will be asked 3 different questions. If she answers all the questions, she will receive an excellent mark; if she answers 2 questions, she will receive a good mark; and if she answers 0 or 1 question, she will receive an unsatisfactory mark. Find the probability of the following:

1. obtaining an excellent mark,
2. receiving an unsatisfactory mark,
3. passing the exam,
4. passing the exam if she knows the answer to the first question,
5. knowing two out of three questions, given that she passed the exam.

Problem 5

The same as previous, but the 3 questions might coincide.

Problem 6

From a brood of mice, containing two white specimens, four mice are taken at random (without return). The probability that both white mice were taken is twice as likely as the probability that neither was taken. How many mice are there in the brood?

Problem 7

A player picks a spot at random within a region S on a flat surface. S is split into four sections, each covering 50%, 30%, 12%, and 8% of the total S area. If the chosen spot falls into one of these sections, the player wins a prize with probabilities of 0.01, 0.05, 0.20, and 0.50, respectively. The player has now selected a spot and won a prize. Which section of the S area is the most likely location for the chosen spot?

Problem 8

Lord Wile loves to drink whiskey, the amount of alcohol consumed per day is random, but it is known that he can drink n glasses per day with probability $\frac{1}{n!}e^{-1}, n = 0, 1, \dots$ Yesterday his wife Lady Wile, his son Liddell and his butler decided to kill the lord. If he didn't drink whiskey that day, Lady Wile must have killed him; if he drank exactly one glass, Liddell had the task to commit the murder; otherwise the butler had to do it. Lady Wile is twice as likely to resort to poisoning as to strangulation; the butler, in contrast, chooses strangulation with twice the likelihood of poisoning; and Liddell is equally likely to choose any of these methods. Despite all efforts, there is no guarantee that Lord Wile will surely die as a result of any of the attempts to kill him, however, he is three times more likely to become a victim of strangulation than poisoning.
Lord Wile is dead today. What is the probability that the butler killed him?

That's all Folks