Conditional expectation. Continuous distributions

Probability theory

Anton Afanasev

Higher School of Economics

DSBA 221
October 14, 2023

Seminar Overview

1. Quiz
2. Conditional distributions
   - Definition
   - Conditional expectation
3. Continuous distributions
   - Probability density function
   - Continuous cases of discrete entities
   - Problems

Quiz

Joint distribution of 2 random variables is set in the table below:

$X \setminus Y$	0	1	2
0	0.2	0.2	0.4
1	0.05	a	b

1. Find a and b such that random variables X and Y are independent.
2. Find $\mathsf{E}(XY)$ with those values of a and b.

Conditional probability mass function

- Conditional p.m.f. of X, given $Y = y$:

$$\mathsf{P}_{X|Y}(x \mid y) = \frac{\mathsf{P}(X = x \cap Y = y)}{\mathsf{P}(Y = y)} = \frac{\mathsf{P}_{X,Y}(x, y)}{\mathsf{P}_Y(y)}.$$

- If X and Y are independent:

$$\mathsf{P}_{X|Y}(x \mid y) = \frac{\mathsf{P}_{X,Y}(x, y)}{\mathsf{P}_Y(y)} = \frac{\mathsf{P}_X(x) \cdot \mathsf{P}_Y(y)}{\mathsf{P}_Y(y)} = \mathsf{P}_X(x).$$

- $\mathsf{P}_{X|Y}(x \mid y)$ is normalized by 1 for each fixed $Y = y$:

$$\sum_{x \in X} \mathsf{P}_{X|Y}(x \mid y) = 1.$$

Problem 1

There are two independent fair coin tosses. Let random variables X and Y be the following:

1.
$$X - \text{number of heads,}$$
$$Y - \text{indicator of an event that both heads and tails were in experiment.}$$

2.
$$X - \text{number of heads,}$$
$$Y - \text{number of tails.}$$

Construct conditional p.m.f.-s $\mathsf{P}_{X|Y}(x \mid y)$ for those cases.

Problem 2

Consider two random variables X and Y. They both take the values 0, 1 and 2. Joint probabilities for each pair are given by the following:

$Y \setminus X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

1. Calculate marginal distributions, expected values and covariance of X and Y.
2. Calculate covariance of the random variables X and V, where $V = X - Y$.
3. Calculate $\mathsf{E}(X \mid Y = 0)$ and $\mathsf{E}(X \mid V = 1)$.
4. The random variable W has the same marginal distribution as X and the random variable Z has the same distribution as Y. It is also known that W and Z are independent. Write down the table for the joint probabilities of W and Z.

Conditional expectation

- Conditional expectation of X, given $Y = y$:

$$\mathsf{E}(X \mid Y = y) = \sum_{x \in X} x \cdot \mathsf{P}_{X|Y}(x \mid y) = \sum_{x \in X} x \cdot \frac{\mathsf{P}_{X,Y}(x, y)}{\mathsf{P}_Y(y)} =$$
$$= \frac{1}{\mathsf{P}_Y(y)} \sum_{x \in X} x \cdot \mathsf{P}_{X,Y}(x, y).$$

- Basically, we calculate $\mathsf{E}(X \mid Y = y)$ as always, but only for relevant $Y = y$, and proportionally shrink by measure of a new sample space $\mathsf{P}_Y(y)$.
- Complex expectations (LOTUS):

$$\mathsf{E}(f(X) \mid Y = y) = \frac{1}{\mathsf{P}_Y(y)} \sum_{x \in X} f(x) \cdot \mathsf{P}_{X,Y}(x, y).$$

Problem 2

$Y \setminus X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

Table: New sample space $Y = 0$, used to calculate $\mathsf{E}(X \mid Y = 0)$.

$Y \setminus X$	0	1	2
0	0	0.2	0.2
1	0.2	0	0.1
2	0.2	0.1	0

Table: New sample space $V = 1$, used to calculate $\mathsf{E}(X \mid V = 1)$.

Probability density function

- In continuous case (geometric approach) the notion of probability mass function does not make sense:

$$\forall x \in \mathbb{R}: \quad \mathsf{P}(X = x) = 0.$$

- Instead, probability density function p.d.f. of X is defined:

$$f_X(x) = \frac{d}{dx} F_X(x).$$

- By definition of a derivative:

$$f_X(x) = \lim_{\Delta x \to 0} \frac{F_X(x + \Delta x) - F_X(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\mathsf{P}(x < X \leq x + \Delta x)}{\Delta x}.$$

- Thus, for infinitesimal Δx:

$$\mathsf{P}(x < X \leq x + \Delta x) \approx f_X(x) \cdot \Delta x.$$

Properties of p.d.f.

- Density is always positive:

$$\forall x \in \mathbb{R}: \quad f_X(x) \geq 0.$$

- C.d.f. $F_X(x)$ is an antiderivative of $f_X(x)$:

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi) d\xi.$$

- $f_X(x)$ is normalized by 1:

$$\int_{-\infty}^{\infty} f_X(x) dx = 1.$$

- Probabilities via $f_X(x)$:

$$\mathsf{P}(a < X \leq b) = \int_{a}^{b} f_X(x) dx.$$

Continuous cases

Marginals, conditionals and independence

- Marginal p.d.f.-s of X and Y:

$$f_X(x) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dy,$$

$$f_Y(y) = \int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)dx.$$

- Conditional p.d.f. of X, given $Y = y$:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

- X and Y are independent iff:

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y).$$

Continuous cases
Expected value

- Expected value of X:
$$\mathsf{E}(X) = \int\limits_{-\infty}^{\infty} x \cdot f_X(x)dx.$$
- Conditional expectation of X, given $Y = y$:
$$\mathsf{E}(X \mid Y = y) = \frac{1}{f_Y(y)} \int\limits_{-\infty}^{\infty} x \cdot f_{X,Y}(x,y)dx.$$
- Law of the unconscious statistician (LOTUS):
$$\mathsf{E}(g(X)) = \int\limits_{-\infty}^{\infty} g(x) \cdot f_X(x)dx,$$
$$\mathsf{E}(g(X,Y)) = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} g(x,y) \cdot f_{X,Y}(x,y)dydx,$$

Problem 3

The p.d.f. of Y is $g(y) = d \cdot y^{-4}, 1 < y < \infty$.

1. Find d.
2. Find c.d.f.
3. Find $\mathsf{E}(Y)$.
4. Find m such that $\mathsf{P}(Y > m) = 0.5$.
5. Find $\mathsf{P}\left(Y > \mathsf{E}(Y)\right)$.

Problem 3

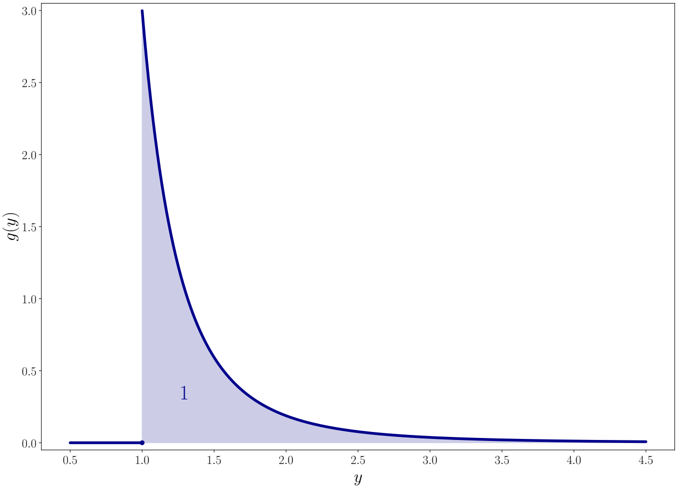

Figure: Probability density function $g(y)$.

Problem 3

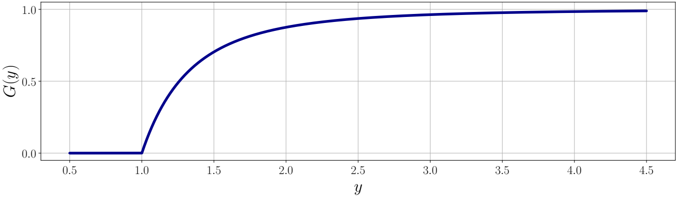

Figure: Cumulative distribution function $G(y)$.

Problem 3

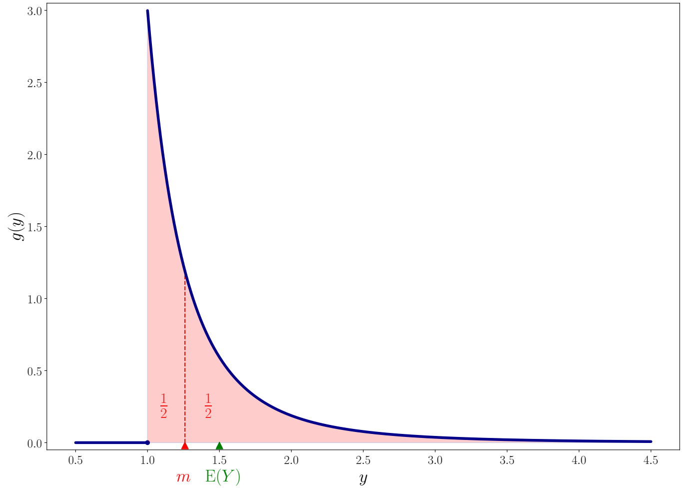

Figure: Median m in p.d.f. $g(y)$.

Problem 3

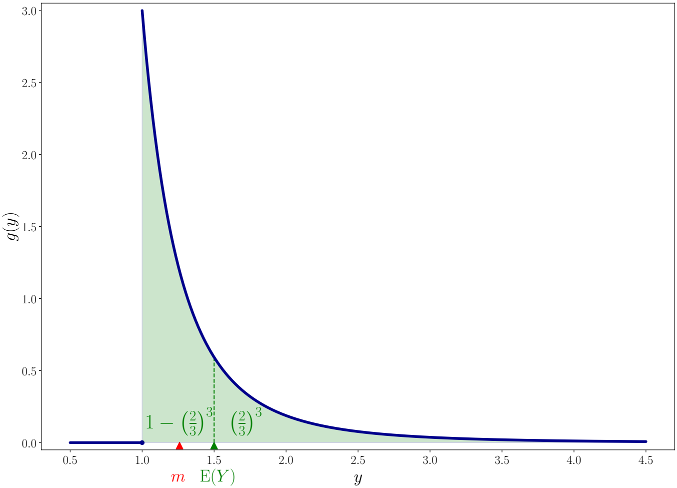

Figure: $\mathsf{P}\left(Y > \mathsf{E}(Y)\right)$ in p.d.f. $g(y)$.

Problem 3

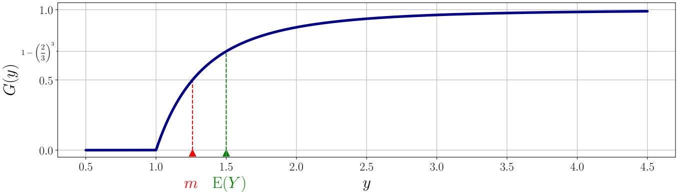

Figure: Median m and mean $\mathsf{E}(Y)$ in c.d.f. $G(y)$.

Problem 4

Let X be a random variable with uniform distribution on the interval $(-2, 3)$. Find $\mathsf{P}(X > 2 \mid X > 1)$.

Problem 4

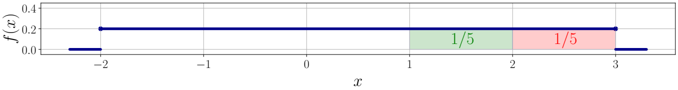

Figure: Probability density function of X.

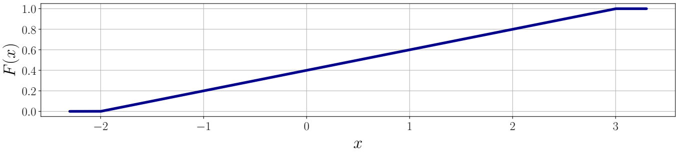

Figure: Cumulative distribution function of X.

Uniform distribution

- Uniform distribution of X from a to b $\iff$ $X \sim \mathcal{U}(a, b)$.
- P.d.f.:
$$f_X(x) = \frac{1}{b-a} \cdot I_{\{a \leq x \leq b\}}.$$
- C.d.f.:
$$F_X(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$
- Mean:
$$\mathsf{E}(X) = \frac{a+b}{2}.$$
- Variance:
$$\mathsf{V}(X) = \frac{(b-a)^2}{12}.$$

That's all Folks