Joint continuous distributions

Probability theory

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Seminar Overview

1. Quiz
2. Correlations revision
3. Joint continuous distributions
4. Complex distributions
   - Composite distributions
   - Variables transformations

Quiz

1. What was the hardest problem in the Midterm?
2. How would you estimate overall difficulty of the Midterm?

Problem 1

Suppose $X \sim \text{Bernoulli}\left(\frac{1}{2}\right)$, $Y \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ and $\rho(X, Y) = 0.8$.
Find the joint distribution of X and Y.

Problem 2

Suppose random variables X and Y have joint normal distribution.

1. If X and Y are standard normal, and $\mathsf{P}(X + Y > 1.96) = 0.025$, what is the correlation between X and Y?
2. If X and Y are independent, what is $\mathsf{P}(X > 1.96 \mid |Y| > 1.96)$?

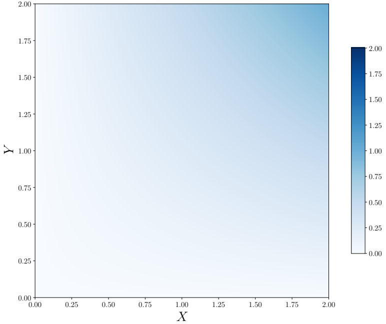
Problem 3

Let two random variables have joint p.d.f.:

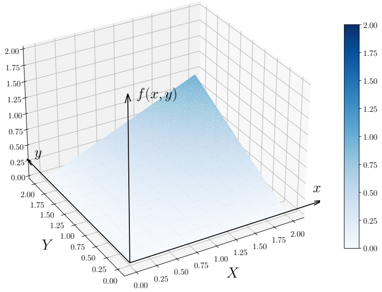
$$f(x,y) = \begin{cases} cxy, & 0 < x, y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

1. Find c, marginal p.d.f. f_X, marginal p.d.f. f_Y. Are X and Y independent?
2. Find $\mathsf{Cov}(X, Y)$.
3. Let $g(x) = \mathsf{E}(Y \mid X = x)$. Find $g(x)$.
4. Find $\mathsf{P}\left(X^2 > Y^2\right)$.

Problem 3



(a) top view



(b) side view

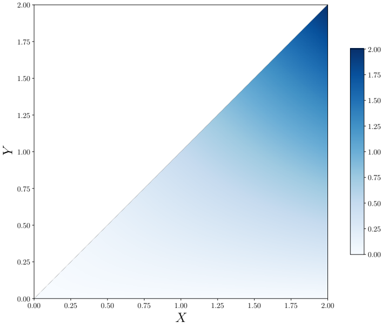
Figure: P.d.f. $f(x,y) = \frac{1}{4}xy \cdot I_{\{0<x,y<2\}}$.

Problem 4

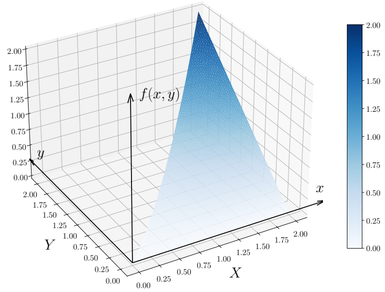
Same as in **Problem 3**, but joint p.d.f. is:

$$f(x,y) = \begin{cases} cxy, & 0 < y < x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4



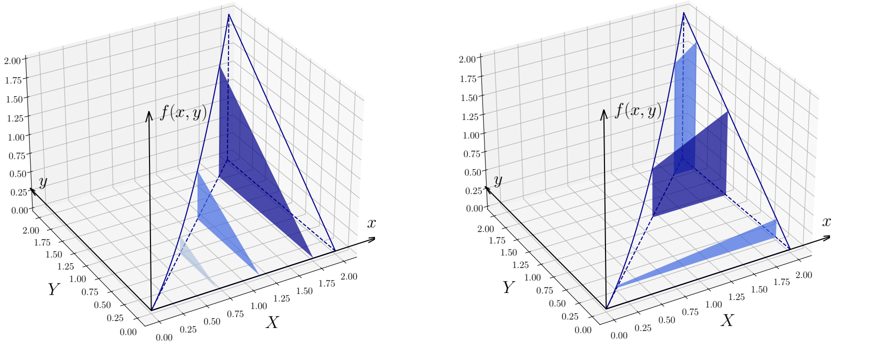
(a) top view



(b) side view

Figure: P.d.f. $f(x,y) = \dfrac{1}{2}xy \cdot I_{\{0<y<x<2\}}$.

Problem 4



(a) X axis shows monotonic increase

(b) Y axis has maximum in $y = 2/\sqrt{3}$

Figure: Marginal slices.

Problem 5

X is a random variable with p.d.f.

$$f(x) = \begin{cases} 0, & x \leq 0, \\ \frac{1}{4}, & 0 < x \leq 1, \\ ax - a, & 1 < x \leq 2, \\ \frac{1}{4}, & 2 < x \leq 3, \\ 0, & x > 3. \end{cases}$$

1. Find c.d.f.
2. Find $\mathsf{P}\left(X > \frac{3}{2}\right)$.
3. Find $\mathsf{P}\left(X < \frac{5}{2} \,\middle|\, X > 1\right)$.
4. Find $\mathsf{E}(X)$.

Problem 5

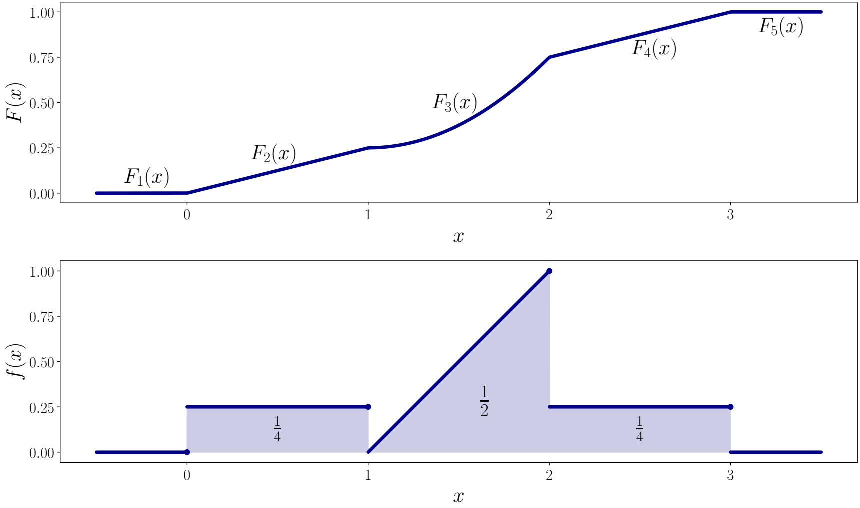


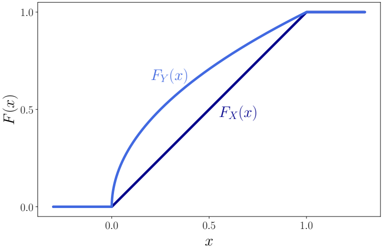
Figure: C.d.f. and p.d.f. of piecewise linear X.

Problem 6

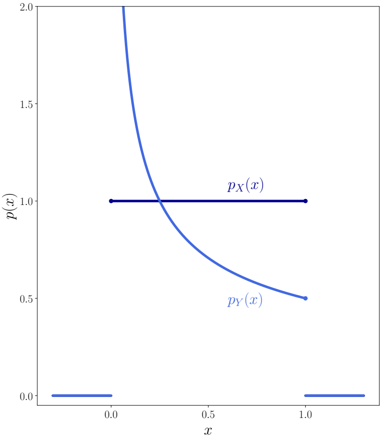
Let X has uniform distribution $\mathcal{U}(0,1)$. $Y = X^2$.

1. Find c.d.f. of Y.
2. Find p.d.f. of Y.

Problem 6



(a) C.d.f.



(b) P.d.f.

Figure: $X \sim \mathcal{U}(0, 1)$ and $Y = X^2$.

Change of variable in p.d.f.

- If $Y = g(X)$, where function g is strictly monotonic, then p.d.f.-s:

$$f_Y(y) = \left|\frac{dg^{-1}(y)}{dy}\right| \cdot f_X(g^{-1}(y)),$$

where absolute value is required for strictly decreasing g, since in that case $F_Y(y) = 1 - F_X(g^{-1}(y))$.

Example

- Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Z = \dfrac{X-\mu}{\sigma} = g(X)$. $X = g^{-1}(Z) = \mu + \sigma Z$.
- Applying change of variable formula:

$$f_Z(z) = \left|\frac{d(\mu+\sigma z)}{dz}\right| \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\mu+\sigma z-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

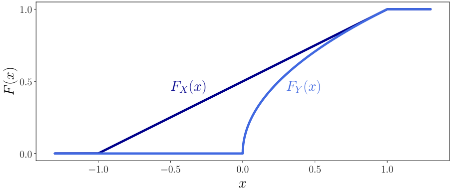
- Usable only if supports of X and $g(X)$ are identical.

Problem 7

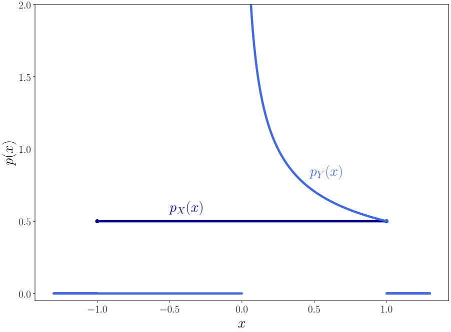
Let X has uniform distribution $\mathcal{U}(-1, 1)$. $Y = X^2$.

1. Find c.d.f. of Y.
2. Find p.d.f. of Y.

Problem 7



(a) C.d.f.



(b) P.d.f.

Figure: $X \sim \mathcal{U}(-1, 1)$ and $Y = X^2$.

Problem 8

Let X, Y are independent random variables, distributed uniformly on $[0, 2]$. Let $W = \max\{X, Y\}, L = \min\{X, Y\}$.

1. Find c.d.f. of W.
2. Find p.d.f. of W.
3. Find $\mathsf{E}(W), \mathsf{P}(W < \mathsf{E}(W))$.
4. Find $\mathsf{E}(L)$.

Problem 8

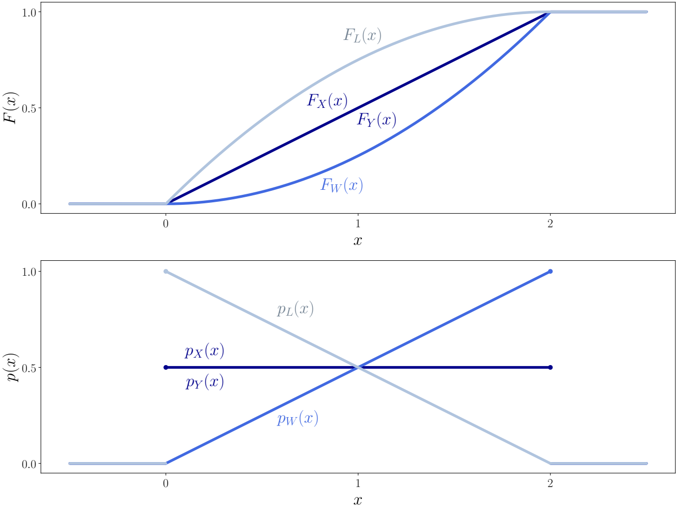


Figure: C.d.f. and p.d.f. of X, Y, $W = \max(X, Y)$ and $L = \min(X, Y)$.

Problem 9

Let X be a random variable with uniform distribution on the interval $[0, \pi]$. Find p.d.f. of random variables:

1. $Y = \sin X$,
2. $Z = X^3$.

Problem 9

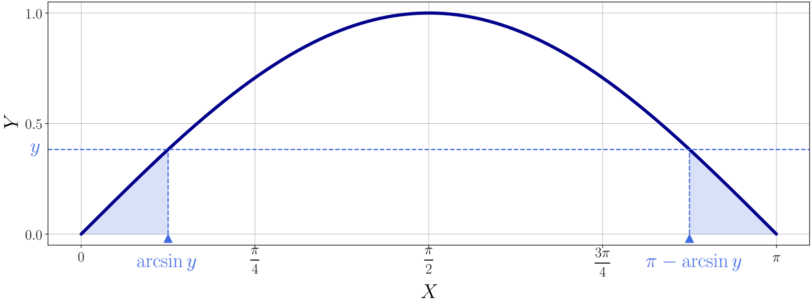


Figure: Calculation of $\mathsf{P}(\sin X \leq y)$.

Problem 9

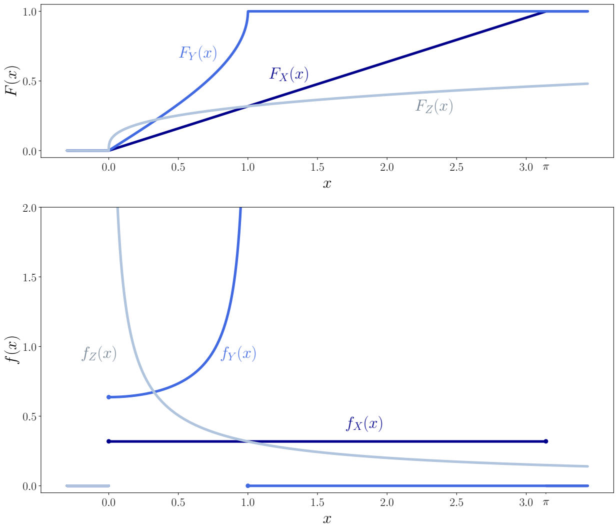


Figure: C.d.f. and p.d.f. of $X \sim \mathcal{U}(0, \pi)$, $Y = \sin X$ and $Z = X^3$.

Problem 10

Suppose there are three assets with returns X_1, X_2, and X_3. It is known that the returns are uncorrelated and their means and standard deviations are:

$$\mu_1 = 0.10, \mu_2 = 0.05, \mu_3 = 0.02,$$

$$\sigma_1 = 0.40, \sigma_2 = 0.20, \sigma_3 = 0.05.$$

Find the "optimal" portfolio with mean $\mu = 0.06$ ("optimal" means smallest variance).

That's all Folks